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An Essay on Free Products of Groups with Amalgamations

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AN ESSAY ON FREE PRODUCTS OF GROUPS WITH AMALGAMATIONS

By B. H. NEUMANN

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CONTENTS

Free products of groups with amalgamated subgroups, first introduced by Schreier (1927) and generalized by Hanna Neumann (1948), are here redefined, studied and applied to a number of problems in abstract group theory. The applications are mainly concerned with existence and embedding theorems. The first four chapters are self-contained, requiring little beyond the fundamentals of abstract group theory. Only in the last chapter is use made of theorems (Gruschko 1940; Kurosch 1934; Hanna Neumann 1949) which are not proved here.

Introduction

Since 1944, when the important book *Teoriya Grupp* by Kuroš appeared, the theory of groups has made great advances, many of them in the directions indicated by Kuroš in the concluding section of his book. When a German translation of the book was being prepared, it was therefore decided to add as an appendix an exposition of some of the recent advances. It fell to me to write this appendix, and the present essay has grown out of it.

In fact the greater part of this essay is a translation of the 'Anhang' which is shortly to appear with Kurosch's *Gruppentheorie* (translated by W. Hahn: Akademieverlag Berlin; a Hungarian translation by A. Gascályi, also with the appendix, is to be published by Akadémiai Kiadó Budapest). I have, however, used this opportunity to revise and supplement it in a number of places, to embody some even more recent results and to draw attention to others; some of these are now in course of publication.

Free products of groups with amalgamated subgroups, which form the central theme of this exposition, are barely mentioned in the book by Kuroš (1944); Magnus (1931, 1932) had early recognized their value, but they had not yet fully established their usefulness. However, they have recently shown themselves to be a powerful and versatile tool, as well as interesting in their own right. They are here introduced, studied and applied to a number of problems in abstract group theory. Much of their structure theory and many of their most recent applications will, however, not be found here; a number of papers not referred to in the text are listed at the end.

Chapter I. The generalized free product; normal forms

1. *Definition and fundamental properties*

In this chapter we start from a given group and explain what it means that it is the *generalized free product* of certain subgroups; and we examine the most important special case in some detail.

Let P be a group and $\mathfrak{E}$ a set of generators of P. Let $\mathfrak{E}$ be the union of subsets $\mathfrak{E}_\alpha$, not necessarily disjoint, where α ranges over an index set A:

$$\mathfrak{E} = \bigcup_{\alpha \in \mathrm{A}} \mathfrak{E}_\alpha.$$

Every set $\mathfrak{E}_\alpha$ generates a subgroup $G_\alpha = \{\mathfrak{E}_\alpha\}$ of P, and P is evidently generated by the groups G_α. Now let $\mathfrak{R}_\alpha$ denote a system of defining relations of G_α; thus $\mathfrak{R}_\alpha$ consists of relations

$$r_{\alpha\rho}(\ldots, e_\alpha, \ldots) = 1 \quad (\rho \in \mathrm{P}_\alpha),$$

where the generators which enter these relations belong to $\mathfrak{E}_\alpha$. If now all these relations together,

$$\mathfrak{R} = \bigcup_{\alpha \in \mathrm{A}} \mathfrak{R}_\alpha,$$

form a set of defining relations of P, then we call P the *generalized free product* (Hanna Neumann 1948) of the subgroups G_α.

Different sets $\mathfrak{E}_\alpha$, $\mathfrak{E}_\beta$ of generators have not been assumed disjoint; hence different groups G_α, G_β can have a non-trivial intersection

$$G_\alpha \cap G_\beta = H_{\alpha\beta} \quad (= H_{\beta\alpha}).$$

If all these intersections are trivial, $H_{\alpha\beta} = \{1\}$ $(\alpha, \beta \in \mathrm{A}, \alpha \neq \beta)$, then P is called simply the *free product*, or, to emphasize the distinction, the *ordinary free product*, of the G_α. An example of a generalized free product which is not an ordinary free product will be found at the end of this chapter.

Our definition does not depend on the 'factors' G_α only, but also on their systems $\mathfrak{E}_\alpha$ of generators and $\mathfrak{R}_\alpha$ of relations. To show that the generators and relations do not enter the definition essentially, we first show that P is in a certain sense (cf. Bates 1947) the *freest* group of its kind.

Theorem. *Let P be the generalized free product of the subgroups G_α, $\alpha \in \mathrm{A}$, and let Q be a group which contains to every G_α a homomorphic copy*†

$$G'_\alpha = G_\alpha \phi_\alpha$$

in such a way that every two homomorphisms ϕ_α, ϕ_β agree where both are defined; thus if

$$h \in H_{\alpha\beta} = G_\alpha \cap G_\beta,$$

then we require $$h\phi_\alpha = h\phi_\beta.$$

Then all the ϕ_α can be extended simultaneously to a homomorphism ϕ of P on to the subgroup of Q generated by the G'_α; that is to say, there is a homomorphic mapping of P into Q which on every G_α agrees with ϕ_α. (1·1)

Proof. We first define ϕ for the generators in $\mathfrak{E}$ by putting, for every $e \in \mathfrak{E}_\alpha$,

$$e\phi = e\phi_\alpha.$$

This defines $e\phi$ uniquely; because if e belongs to more than one $\mathfrak{E}_\alpha$, then all the corresponding ϕ_α assign it the same map in Q. Now any relation

$$r(\ldots, e_\sigma, \ldots, e_\tau, \ldots) = 1$$

between generators in $\mathfrak{E}$ follows from certain defining relations of the G_α; the same relations hold also between the corresponding generators of the G'_α, as the mappings ϕ_α are homomorphisms, and the relation

$$r(\ldots, e_\sigma\phi, \ldots, e_\tau\phi, \ldots) = 1$$

between them then follows. Hence the mapping ϕ generates a homomorphism of P on to that subgroup of Q which is generated by $\mathfrak{E}\phi$, that is, by the G'_α; and as ϕ maps the system $\mathfrak{E}_\alpha$ of generators of G_α exactly as ϕ_α does, the homomorphisms ϕ and ϕ_α agree on G_α. This completes the proof of the theorem.

It follows from this theorem that P is uniquely determined (apart from isomorphism) by the subgroups G_α and their intersections $H_{\alpha\beta}$. For let P' be another group which is the generalized free product of subgroups G'_α $(\alpha \in \mathrm{A})$, and let equally indexed groups G_α and G'_α be isomorphic; let there be given to each α an isomorphic mapping ϕ_α of G_α on to G'_α which maps the intersections $H_{\alpha\beta}$ on to the corresponding intersections $H'_{\alpha\beta} = G'_\alpha \cap G'_\beta$. Specifically, if $h \in H_{\alpha\beta}$, we assume

$$h\phi_\alpha = h\phi_\beta,$$

† Mappings (such as homomorphisms, endomorphisms, permutations, etc.) are written as right-hand operators.

and if $h' \in H'_{\alpha\beta}$, we assume conversely

$$h'\phi_\alpha^{-1} = h'\phi_\beta^{-1}.$$

Then there is a homomorphism ϕ of P into P' which in every G_α coincides with ϕ_α; and there is also a homomorphism ψ of P' into P which in every G'_α coincides with ϕ_α^{-1}. The product $\phi\psi$ is thus an endomorphism of P which maps each G_α according to $\phi_\alpha\phi_\alpha^{-1}$, that is, simply identically on to itself. As the G_α together generate P, $\phi\psi$ must be the identical automorphism of P. Similarly, $\psi\phi$ is the identical automorphism of P'. But ϕ and ψ must then be mutually inverse isomorphisms of P on to P' and back, and P and P' are seen to be isomorphic. This shows that the generalized free product does not depend on the systems of generators and relations which enter the definition but only on the systems of groups G_α and intersections $H_{\alpha\beta}$, that is, the so-called *amalgam*† of the G_α.

Every group P has a—rather trivial—representation as a generalized free product. We choose a system of subgroups G_α of P such that every finite set of elements of P is contained in (at least) one of the G_α (thus, for example, the G_α can be all the finitely generated subgroups of P). If $\mathfrak{E}_\alpha$ is a set of generators of G_α, then $\mathfrak{E} = \bigcup_{\alpha\in \mathrm{A}}\mathfrak{E}_\alpha$ is evidently a set of generators of P. Any relation of P involves a finite number of elements only: these lie in a G_α, and in this the relation must be satisfied; thus it follows from the defining relations of G_α, and we see that all defining relations of all the G_α together form a set of defining relations of P. Thus P is the generalized free product of the G_α. If P is finitely generated, then it occurs itself among these G_α; but even if it does not (and so cannot be finitely generated), it coincides with the amalgam, that is, the set-theoretical union of the subgroups G_α. We shall call P the *proper* generalized free product of subgroups G_α if it is the generalized free product but does not coincide with the amalgam of the G_α.

2. *One amalgamated subgroup; the normal form*

In order to obtain useful results one has to impose some rather restrictive conditions upon the groups; this we now proceed to do.

A central place in the theory is occupied by the special case first studied by Schreier (1927). Here all the intersections $H_{\alpha\beta}$ coincide to form a single group H, so that

$$G_\alpha \cap G_\beta = H,$$

when $\alpha, \beta \in \mathrm{A}$, $\alpha \neq \beta$. This case arises, for example, whenever A consists of two indices only, in other words, when the product has only two factors. We call P the '(generalized‡) free product of the G_α with the amalgamated subgroup H'. In this, as in the ordinary free product (with only trivial amalgamations), one can represent the elements in a certain *normal form*. The remainder of this chapter is chiefly concerned with this normal form and its principal properties.

We choose in every group G_α a system S_α of left coset representatives modulo H; thus every element $g \in G_\alpha$ is uniquely represented in the form

$$g = sh \quad (s \in S_\alpha,\ h \in H).$$

† Baer (1949); cf. also chapter II, §9.

‡ We use 'free product with amalgamated subgroups' and 'generalized free product' synonymously, the amalgamations being understood.

For the sake of simplicity we take the unit element as the representative of H itself, so that $1 \in S_\alpha$ for every α. Now we distinguish certain words in elements of the G_α; specifically we call

$$w = s_1 s_2 \dots s_n h$$

a *normal word* if it satisfies the following three conditions:

(1) Every *component* s_i $(1 \leqslant i \leqslant n)$ is a representative $\neq 1$ belonging to one of the S_α.

(2) Successive components s_i belong to different systems of representatives; in other words, if $1 \leqslant i < n$, $s_i \in S_\alpha$, $s_{i+1} \in S_\beta$, then $\alpha \neq \beta$.

(3) The last component belongs to the common subgroup,

$$h \in H.$$

We call n the *length* of the normal word. The elements of H, and these only, are normal words of zero length. We denote the set of all normal words by $\mathfrak{W}$.

Every normal word $w \in \mathfrak{W}$ represents† a well-defined element $p \in P$. We shall now show that every element of P is represented by one and only one normal word. To this end we introduce mappings of $\mathfrak{W}$ into itself; to every element $g \in G_\alpha$ we make correspond a mapping σ_g. This mapping σ_g is to map the normal word

$$w = s_1 s_2 \dots s_n h$$

on to a normal word $$w' = w\sigma_g$$
as follows:

(1) If $n > 0$, and if $s_n \in S_\alpha$, that is, if s_n lies in the same group G_α as g, then $s_n hg$ is a certain element of G_α, which we represent in the form

$$s_n hg = s'h' \quad (s' \in S_\alpha,\ h' \in H).$$

Now if $s' \neq 1$, we put $$w' = s_1 s_2 \dots s_{n-1} s'h';$$

if, on the other hand, $s' = 1$, then we omit it and put

$$w' = s_1 s_2 \dots s_{n-1} h'.$$

(2) If $n = 0$, or if $n > 0$ and $s_n \in S_\beta$ with $\beta \neq \alpha$, that is, if s_n is not contained in the same group G_α as g, we represent the element $hg \in G_\alpha$ in the form

$$hg = s'h' \quad (s' \in S_\alpha,\ h' \in H).$$

If now $s' \neq 1$, we put $$w' = s_1 s_2 \dots s_n s'h';$$

if, on the other hand, $s' = 1$, we omit it again, putting

$$w' = s_1 s_2 \dots s_n h'.$$

In this way $w' = w\sigma_g$ is defined for every $w \in \mathfrak{W}$, and one verifies easily that w' is again a normal word. If g is contained in more than one group G_α, thus in their intersection H, then also

$$hg = h' \in H,$$

and one confirms without difficulty that then

$$w' = s_1 s_2 \dots s_n h',$$

† A *word* is a string of symbols; if we interpret it as a product (which is written in the same way) we obtain an element of the group, and we say the word *represents* the element. The same element of the group can be represented by different words.

irrespective of whether one defines it according to (1) or (2), irrespective, that is to say, of whether one considers g as lying in the same group G_α as s_n or in a different one.

Thus σ_g is defined as mapping of $\mathfrak{W}$ into itself. Next we show that if g, g' are two elements of the same group G_α, then

$$\sigma_{gg'} = \sigma_g \sigma_{g'}, \tag{2·1}$$

that is to say, the mapping which belongs to the product gg' is obtained by first carrying out the mapping belonging to g, then that belonging to g'.

Let $\quad w = s_1 s_2 \ldots s_n h, \quad w' = w\sigma_g, \quad w'' = w'\sigma_{g'} \quad$ and $\quad w^* = w\sigma_{gg'}.$

We again distinguish two cases:

(1) If $n>0$, and if $s_n \in S_\alpha$, that is, if s_n belongs to the same group G_α as g and g', then we put

$$s_n hg = s'h' \quad (s' \in S_\alpha,\ h' \in H),$$

and

$$s'h'g' = s''h'' \quad (s'' \in S_\alpha,\ h'' \in H).$$

Then either $w'' = s_1 s_2 \ldots s_{n-1} s'' h''$ or $w'' = s_1 s_2 \ldots s_{n-1} h''$ according as $s'' \neq 1$ or $s'' = 1$. On the other hand, if we put

$$s_n hgg' = s^* h^* \quad (s^* \in S_\alpha,\ h^* \in H),$$

then either $w^* = s_1 s_2 \ldots s_{n-1} s^* h^*$ or $w^* = s_1 s_2 \ldots s_{n-1} h^*$, according as $s^* \neq 1$ or $s^* = 1$. But as

$$s''h'' = s'h'g' = s_n hgg' = s^* h^*,$$

it follows that $s'' = s^*$ and $h'' = h^*$, and thus also $w'' = w^*$. It makes no difference whether $s' \neq 1$ or $s' = 1$, as—in case $n>1$—s_{n-1} can certainly not lie in S_α.

(2) Let $n = 0$, or $n>0$ but $s_n \in S_\beta$ with $\beta \neq \alpha$. Then we put

$$hg = s'h' \quad (s' \in S_\alpha,\ h' \in H)$$

and (again irrespective of whether $s' \neq 1$ or $s' = 1$)

$$s'h'g' = s''h'' \quad (s'' \in S_\alpha,\ h'' \in H).$$

Thus $w'' = s_1 s_2 \ldots s_n s'' h''$ if $s'' \neq 1$, and $w'' = s_1 s_2 \ldots s_n h''$ if $s'' = 1$. On the other hand, putting

$$hgg' = s^* h^* \quad (s^* \in S_\alpha,\ h^* \in H),$$

we have $w^* = s_1 s_2 \ldots s_n s^* h^*$ if $s^* \neq 1$, and $w^* = s_1 s_2 \ldots s_n h^*$ if $s^* = 1$. Again, as before,

$$s''h'' = s'h'g' = hgg' = s^* h^*,$$

so that $s'' = s^*$ and $h'' = h^*$ and finally again $w'' = w^*$. Thus in both cases (2·1) has been proved.

Equation (2·1) means that the σ_g which correspond to the elements $g \in G_\alpha$ represent the group G_α homomorphically; in other words, if one maps each $g \in G_\alpha$ on to σ_g and denotes this mapping by ϕ_α, so that for all $g \in G_\alpha$

$$g\phi_\alpha = \sigma_g,$$

then ϕ_α is a homomorphism. It follows that for fixed α the σ_g corresponding to $g \in G_\alpha$ form a group themselves:

$$G_\alpha \phi_\alpha = \Sigma_\alpha,$$

say. Therefore every mapping σ_g has a right and left inverse, hence is a permutation of $\mathfrak{W}$.

Let Σ denote the group of permutations of $\mathfrak{W}$ which is generated by the Σ_α $(\alpha \in \mathrm{A})$. Then we prove easily that Σ is transitive on $\mathfrak{W}$; it suffices to show that Σ permutes a single normal word, say $w_0 = 1$, into every normal word. Now if

$$w = s_1 s_2 \dots s_n h$$

is a normal word, then so are the words

$$w_1 = s_1 1,$$
$$w_2 = s_1 s_2 1,$$
$$\dots\dots\dots\dots\dots\dots$$
$$w_n = s_1 s_2 \dots s_n 1,$$

and furthermore

$$w_1 = w_0 \sigma_{s_1},$$
$$w_2 = w_1 \sigma_{s_2},$$
$$\dots\dots\dots\dots$$
$$w_n = w_{n-1} \sigma_{s_n},$$
$$w = w_n \sigma_h.$$

Thus $w = w_0 \sigma$, where $\sigma = \sigma_{s_1} \sigma_{s_2} \dots \sigma_{s_n} \sigma_h$ is in Σ.

The group Σ and the homomorphisms ϕ_α $(\alpha \in \mathrm{A})$ together satisfy the conditions of theorem (1·1); for ϕ_α maps G_α homomorphically on to $\Sigma_\alpha \subseteq \Sigma$; and where two such mappings ϕ_α and ϕ_β are defined simultaneously, namely, in H, there they agree—we had seen that for $g \in H$ the mapping σ_g does not depend on which group G_α the element g is assigned to. Thus theorem (1·1) shows that the ϕ_α can be extended simultaneously to a homomorphism ϕ of P on to the group generated by the $G_\alpha \phi_\alpha$—that is, on to Σ itself.

Next we want to show that ϕ is an isomorphism, hence that Σ is isomorphic to P. This requires two preparatory lemmas.

LEMMA. *If $w \in \mathfrak{W}$ is a normal word which represents the element $p \in P$, and if $g \in G_\alpha$, then the normal word*

$$w' = w\sigma_g$$

represents the element $pg \in P$. (2·2)

Proof. This results immediately from the definition of σ_g; for if $w = s_1 s_2 \dots s_n h$, then

$$w' = s_1 s_2 \dots s_{n-1} s'h', \quad \text{where} \quad s'h' = s_n hg \text{ in } P; \qquad (a)$$

or

$$w' = s_1 s_2 \dots s_{n-1} h', \quad \text{where} \quad h' = s_n hg \text{ in } P; \qquad (b)$$

or

$$w' = s_1 s_2 \dots s_n s'h', \quad \text{where} \quad s'h' = hg \text{ in } P; \qquad (c)$$

or

$$w' = s_1 s_2 \dots s_n h', \quad \text{where} \quad h' = hg \text{ in } P; \qquad (d)$$

thus in any case w' represents the product pg if p is the element represented by w.

LEMMA. *If $w \in \mathfrak{W}$ is a normal word which represents the element $p \in P$, if furthermore $q \in P$ and $q\phi = \sigma \in \Sigma$, then the normal word $w^* = w\sigma$ represents the product $pq \in P$.* (2·3)

Proof. This extension of lemma (2·2) follows from it by induction over the number of factors in a representation of q as a product of elements of the G_α. Let $q = q'g$, where q' can be written as a product of fewer factors in the G_α than q, and where g itself is contained in a G_α. If $q'\phi = \sigma'$, then $q\phi = \sigma = \sigma' \sigma_g$ because ϕ is a homomorphism and because $g\phi = \sigma_g$.

We may assume that $w' = w\sigma'$ represents the product pq' (by the induction hypothesis). But then $w^* = w'\sigma_g = w\sigma$ represents the product $pq'g = pq$, by lemma (2·2), and lemma (2·3) follows.

It follows from this lemma that every element $p \in P$ is represented by at least one normal word $w \in \mathfrak{W}$. For the normal word $w_0 = 1$ evidently represents the unit element $1 \in P$; and if $\sigma = p\phi$, then the normal word $w = w_0\sigma$ represents the product $1p = p \in P$, by lemma (2·3).

Moreover, it follows that ϕ is in fact an isomorphism. For if $p_0 \in P$ is contained in the kernel of ϕ, that is, if $p_0\phi$ is the unit element ι of Σ—the identical permutation of $\mathfrak{W}$—and if $w \in \mathfrak{W}$ represents the element $p \in P$, then $w\iota = w$ represents the product pp_0. But a word w can only represent a single element of P; hence $pp_0 = p$ and $p_0 = 1$.

Theorem (Schreier 1927). *Every element $p \in P$ is represented by one and only one normal word $w \in \mathfrak{W}$.* (2·4)

Proof. We have already seen that every $p \in P$ is represented by at least one normal word; it only remains to show that p cannot be represented by more than one normal word. If now $\sigma = p\phi$, then p is represented by $w = w_0\sigma$, where again $w_0 = 1 \in \mathfrak{W}$. Let $w' \in \mathfrak{W}$ be another normal word which represents the same element p, and let $w' = w_0\sigma'$—such a σ' exists because of the transitivity of Σ; finally, let p' be the original of σ' under ϕ, so that $\sigma' = p'\phi$; then w' represents p', and thus $p' = p$, $\sigma' = \sigma$ and $w' = w$. This proves the theorem.

The uniquely determined normal word representing p we call the *normal form* of p. Because of the theorem we may identify the elements of p with their normal forms. Then Σ becomes a group of permutations of the elements of P; in fact, the elements of Σ are just the *right multiplications* of P; with fixed $\sigma = q\phi$ and for all $p \in P$ one has

$$p\sigma = pq.$$

Σ is called the *regular* permutation representation of P. It is well known that right multiplications and the regular permutation representation can be defined in the same way for arbitrary groups, not only for free products with an amalgamated subgroup.†

3. *Computation of the normal form; uniqueness of the length*

The normal form of an element $p \in P$ can be computed without difficulty. Let the element be given in the form

$$p = g_1 g_2 \cdots g_n, \tag{3·1}$$

where the g_i lie in the groups G_α. If two successive factors g_i, g_{i+1} belong to the same group G_α, they can be combined to a single such factor. We may therefore assume that in (3·1) no two successive factors lie in the same group G_α. If there is more than one factor, that is, $n > 1$, then none of the g_i belongs to the intersection H. Now we start with g_1: if $n = 1$ and $g_1 \in H$, then $p = g_1$ is already the normal form. If not, then g_1 is contained in a unique group G_α, and not in H. We put

$$g_1 = s_1 h_1 \quad (s_1 \in S_\alpha,\ h_1 \in H);$$

then $s_1 \neq 1$. Now

$$p = s_1 h_1 g_2 \cdots g_n = s_1 g_2' g_3 \cdots g_n,$$

where $g_2' = h_1 g_2$ lies in the same group $G_\beta \neq G_\alpha$, as g_2. Next we put

$$g_2' = s_2 h_2 \quad (s_2 \in S_\beta, h_2 \in H);$$

† Cf., for example, Kuroš (1944; 1953), §5.

here again $s_2 \neq 1$, as otherwise g_2' and thus also g_2 would be in H. Now

$$p = s_1 s_2 h_2 g_3 \dots g_n = s_1 s_2 g_3' g_4 \dots g_n,$$

where $g_3' = h_2 g_3$ lies in the same group G_γ as g_3. We continue in the same way until after n steps we arrive at the normal form

$$p = s_1 s_2 \dots s_n h.$$

The normal form we have introduced depends not only on the groups G_α and H but also on the systems S_α of left coset representatives which we have chosen in G_α. But the method for finding the normal form of the elements of P, which we have just described, shows that the length of the normal form does not depend on the choice of the S_α but only on the element of the group itself; we may therefore call it the *length of the element*:

Corollary. *If* $n > 1$, *if* $$p = g_1 g_2 \dots g_n,$$ *and if no two successive factors* g_i, g_{i+1} *are elements of the same group* G_α, *then* n *is the length of* p. *If* $n = 1$, *the length of* p *is* 0 *or* 1 *according as* p *lies in* H *or not.* (3·2)

The process we have described for computing the normal form of an element of P does not depend on the fact that P is the generalized free product of the G_α. If P is an arbitrary group containing subgroups G_α such that any two different such subgroups intersect in a fixed group H, then we can choose systems S_α of left coset representatives of the G_α modulo H and define normal words as before. If the element $p \in P$ belongs to the subgroup of P generated by all the G_α—and only then—it can be written in the form

$$p = g_1 g_2 \dots g_n,$$

with each g_i lying in a group G_α, and no two successive ones in the same. The process described above then assigns to p a well-defined normal word (of length n, if $n > 1$; of length 1 or 0 if $n = 1$)

$$w = s_1 s_2 \dots s_n h,$$

which represents p. If P is not generated by the G_α, then there will be elements of P which are not represented by any normal word; and if P is not the generalized free product of the G_α then there may be elements represented by more than one normal word. In fact we can prove a converse of theorem (2·4):

Theorem. *Let the group* P *have subgroups* G_α $(\alpha \in \mathrm{A})$ *which intersect pairwise in a common subgroup* H,

$$G_\alpha \cap G_\beta = H \quad (\alpha, \beta \in \mathrm{A},\ \alpha \neq \beta).$$

If every element $p \in P$ *is represented by at least one normal word (as defined above) and if normal words of different lengths represent different elements of* P, *then* P *is the generalized free product of the* G_α, *and thus in particular every element is represented by a unique*† *normal word.* (3·3)

Proof. The subgroups G_α generate P, as every element of P is assumed to be represented by a normal word. As generators of G_α we choose all its elements; the defining relations can be taken, one for every ordered pair of elements of G_α, to state which element of G_α is their product. We take an arbitrary relation in P; this can be written in the form

$$g_1 g_2 \dots g_n = 1, \tag{3·4}$$

† In terms of a fixed choice of the systems S_α.

with each g_i belonging to a G_α. If here no successive factors belong to the same group G_α, then the left-hand side is represented by a normal word of length n, if $n > 1$; but the right-hand side is represented by $w_0 = 1$ of length 0. This would contradict our assumptions; hence $n = 1$, and (3·4) is a relation in a G_α. Alternatively, if $n > 1$, then there must be two successive factors g_i, g_{i+1} in the same G_α. These can be replaced by a single factor, using the defining relations of that G_α. This reduces the number of factors. An easy induction over this number then shows that the relation (3·4) follows from the defining relations of all the G_α. Hence P is their generalized free product, and the theorem follows.

4. *Some special subgroups*

We use this theorem, together with the freedom we still have in choosing the systems S_α of left coset representatives, to describe some special subgroups of a generalized free product.

THEOREM (Hanna Neumann 1948). *Let P be the free product of groups G_α ($\alpha \in$ A) with an amalgamated subgroup H. In every G_α let there be given a subgroup A_α which intersects H in a fixed subgroup B,*

$$A_\alpha \subseteq G_\alpha, \quad A_\alpha \cap H = B.$$

Then the subgroup M, say, of P generated by the A_α is their generalized free product (with amalgamated B). If, in particular, the subgroups A_α have trivial intersection with H, then they generate their ordinary free product. (4·1)

Proof. In every A_α we choose a system T_α of left coset representatives modulo B, representing B itself by the unit element. Then no two different representatives t, t' in T_α lie in the same left coset of G_α modulo H, as otherwise $t^{-1}t'$ would lie in $A_\alpha \cap H = B$. We can therefore choose the elements of T_α to represent the left cosets of G_α modulo H in which they lie, and then complete the system S_α by choosing representatives also for the remaining left cosets of G_α modulo H. Thus $T_\alpha \subseteq S_\alpha$ for every $\alpha \in$ A. Now if m is an element of the group M generated by the A_α, then there is a normal word (relating to M and its subgroups A_α)

$$w = t_1 t_2 \ldots t_n b$$

which represents it; here every t_i is different from 1 and belongs to a T_α, no two successive t_i, t_{i+1} to the same, and $b \in B$. But w is also a normal word if considered in relation to P and its subgroups G_α, and as such it is unique by theorem (2·4). By theorem (3·3) then M is the generalized free product of the A_α, and the theorem follows.

Although this theorem deals only with a very restricted class of subgroups of P, it will nevertheless prove useful for various applications. The following theorem deals with an even more restricted situation.

THEOREM. *Let P be the generalized free product of two groups A and B, and let their intersection be*

$$H = A \cap B.$$

Let a set of elements t_α (where α ranges over an index set A) be given with the following two properties:

(i) *Every t_α belongs to the normalizer of H in B, that is to say, $t_\alpha \in B$ and $t_\alpha H = Ht_\alpha$.*

(ii) *No two different elements t_α, t_β ($\alpha \neq \beta$) lie in the same left coset (or, what amounts to the same because of (i), right coset) modulo H, that is,*

$$t_\alpha H \neq t_\beta H.$$

Then the subgroup Q *of* P *generated by the groups*

$$G_\alpha = t_\alpha^{-1} A t_\alpha$$

is the generalized free product of these G_α, *with*

$$H = G_\alpha \cap G_\beta \quad (\alpha \neq \beta)$$

amalgamated. (4·2)

Proof. We first show that different groups G_α, G_β intersect in just H. Clearly each G_α contains H, by the first assumption on the t_α; hence

$$H \subseteq G_\alpha \cap G_\beta.$$

Let now $g \in G_\alpha \cap G_\beta$; then

$$g = t_\alpha^{-1} a t_\alpha = t_\beta^{-1} a' t_\beta \quad (a, a' \in A).$$

This gives

$$t_\alpha^{-1} a^{-1} t_\alpha t_\beta^{-1} a' t_\beta = 1.$$

Now $t_\alpha t_\beta^{-1}$ is in B but not in H, because of (ii). If a and a' were not in H, the left-hand side of our equation would have length $\geqslant 4$ and could not be the unit element. Hence at least one of a, a' is in H, and then g is in H because of (i). (In fact then a, a' are both in H.) It follows that

$$G_\alpha \cap G_\beta \subseteq H,$$

and thus

$$G_\alpha \cap G_\beta = H.$$

Now let S be a set of left coset representatives of A modulo H. Thus every element $a \in A$ is uniquely of the form

$$a = sh \quad (s \in S, \ h \in H).$$

We put

$$S_\alpha = t_\alpha^{-1} S t_\alpha.$$

Then S_α is a set of left coset representatives of G_α modulo H $(= t_\alpha^{-1} H t_\alpha)$. Every element q of Q is generated by elements of the G_α and hence is represented by at least one normal word

$$s_{\alpha(1)} s_{\alpha(2)} \dots s_{\alpha(n)} h \quad (s_{\alpha(i)} \in S_{\alpha(i)}, \ h \in H)$$

in Q. Writing

$$s_{\alpha(i)} = t_{\alpha(i)}^{-1} \hat{s}_i t_{\alpha(i)}$$

with $\hat{s}_i \in S$, we express q as an element of P:

$$q = t_{\alpha(1)}^{-1} \hat{s}_1 t_{\alpha(1)} t_{\alpha(2)}^{-1} \hat{s}_2 t_{\alpha(2)} \dots t_{\alpha(n)}^{-1} \hat{s}_n t_{\alpha(n)} h.$$

Here no $\hat{s}_i$ is the unit element, nor even in H, and successive $s_{\alpha(i)}$ belong to different groups G_α, that is, $\alpha(i) \neq \alpha(i+1)$ $(i = 1, \dots, n-1)$. Hence no $t_{\alpha(i)} t_{\alpha(i+1)}^{-1}$ lies in H. Thus the successive factors

$$\hat{s}_1, t_{\alpha(1)} t_{\alpha(2)}^{-1}, \hat{s}_2, \dots, t_{\alpha(n-1)} t_{\alpha(n)}^{-1}, \hat{s}_n$$

are alternately out of A and B, and none of them in H. The first factor $t_{\alpha(1)}^{-1}$ and the last $t_{\alpha(n)} h$ may or may not be in H; if one or the other is not in H, then it is in B, and carries the alternation further. The normal form of q in P then has length $2n-1$ or $2n$ or $2n+1$, according as it begins and ends with a factor in A, or begins or ends with a factor in A but ends or begins with a factor in B, or begins and ends with a factor in B respectively. Thus the number n is uniquely determined by the element q; but this number is the length of a normal word representing q in Q. By theorem (3·3) then Q is the generalized free product of the G_α, and the theorem follows.

Corollary.† *Let P be the ordinary free product of A and B and b an element, not the unit element, of B. Then A and $b^{-1}Ab$ generate their ordinary free product in P.* (4·3)

Here the intersection H is trivial, and we have two transforming elements $t_0 = 1$, $t_1 = b$. The assumptions of the theorem are easily verified.

Corollary. *Let P be the free group with two generators a, b. Then the elements $b^{-i}ab^i$, $i = 0, \pm 1, \pm 2, \ldots$ are free generators of the subgroup they generate.* (4·4)

Here P is the free product of $\{a\}$ and $\{b\}$, the subgroup the free product of all $\{b^{-i}ab^i\}$. The result is, of course, well known.

5. *Elements of finite order; an example*

Finally, we prove a theorem on elements of finite order in a free product with an amalgamated subgroup.

Theorem. *Let P be the free product of groups G_α with the amalgamated subgroup H; if p is an element of finite order in P, then p is conjugate to an element in (at least) one of the G_α.* (5·1)

For the proof we require two lemmas. We call the element $p \in P$ *cyclically reduced* if none of its conjugates in P has smaller length than itself. Clearly every element of P has cyclically reduced conjugates.

Lemma. *If p is cyclically reduced and if it has the normal form*

$$p = s_1 s_2 \ldots s_n h$$

of length $n > 1$, then s_1 and s_n belong to different subgroups $G_\alpha \neq G_\beta$. (5·2)

Proof. Assume $n > 1$ but s_1 and s_n both in the same G_α. Then consider

$$p' = s_1^{-1} p s_1 = s_2 \ldots s_n h s_1.$$

As $s_n h s_1 \in G_\alpha$, we can write $\quad s_n h s_1 = s' h' \quad (s' \in S_\alpha,\ h' \in H)$,
and p' has normal form

$$p' = s_2 \ldots s_{n-1} s' h' \quad \text{or} \quad p' = s_2 \ldots s_{n-1} h',$$

according as $s' \neq 1$ or $s' = 1$. In any case p' has length $n-1$; hence p was not cyclically reduced, and the lemma follows.

Lemma. *If p is cyclically reduced and has length $n > 1$, then for every positive integer k the length of p^k is $k.n$; it follows that p then has infinite order.* (5·3)

Proof. In the normal form $\quad p = s_1 s_2 \ldots s_n h$

let the component s_i belong to $G_{\alpha(i)}$. By lemma (5·2) then $\alpha(1) \neq \alpha(n)$. Assume it has been shown already that the normal form of p^{k-1} is

$$p^{k-1} = s'_1 s'_2 \ldots s'_{(k-1)n} h',$$

and that $s'_{(k-1)n} \in S_{\alpha(n)}$. We calculate the normal form of p^k by determining successively $s''_1, s''_2, \ldots, s''_n$ and h'' such that

$$h' s_1 = s''_1 h^{(1)} \quad (s''_1 \in S_{\alpha(1)},\ h^{(1)} \in H)$$

$$h^{(1)} s_2 = s''_2 h^{(2)} \quad (s''_2 \in S_{\alpha(2)},\ h^{(2)} \in H)$$

....

$$h^{(n-1)} s_n = s''_n h'' \quad (s''_n \in S_{\alpha(n)},\ h'' \in H).$$

† This is also an easy corollary of results of Kurosch (1934); cf. Kurosch (1944) or (1953), §44.

Then the normal form of p^k is evidently

$$p^k = s_1' s_2' \dots s_{(k-1)n}' s_1'' s_2'' \dots s_n'' h'',$$

which has length $k.n$, and again $s_n'' \in S_{\alpha(n)}$. Induction over k proves the lemma.

The proof uses only that $n > 1$ and $\alpha(1) \neq \alpha(n)$, and we can derive from it a converse of lemma (5·2).

COROLLARY. *If p has length $n > 1$ and if in its normal form*

$$p = s_1 s_2 \dots s_n h$$

the components s_1 and s_n belong to different groups $G_{\alpha(1)}$ and $G_{\alpha(n)}$, then p is cyclically reduced. (5·4)

For let p' of length n' be a cyclically reduced conjugate of p, say $p' = t^{-1}pt$. Then also $p'^k = t^{-1}p^k t$. Assume first $n' > 1$. Then the length of p'^k is $k.n'$, that of p^k is $k.n$, and that of $t^{-1}p^k t$ can differ from that of p^k at most by twice the length of t. If this is m, we have

$$|k.n' - k.n| \leqslant 2m$$

for all $k > 0$, whence $n' = n$, showing that p has itself minimal length among its conjugates, and thus is cyclically reduced. If $n' \leqslant 1$, then p' lies in some G_α and so then does p'^k for all k. Hence p'^k has length 1 or 0, and we get

$$|1 - k.n| \leqslant 2m \quad (\text{or } |0 - k.n| \leqslant 2m)$$

for all k, which is incompatible with $n > 1$.

Proof of theorem (5·1). Let p be an element of finite order in P, and let p^* be a cyclically reduced conjugate of p. Then p^* also has finite order, and by lemma (5·3) its length cannot exceed 1. Hence p^* lies in (at least) one of the factors G_α, and the theorem follows.

COROLLARY. *The free product of locally infinite groups with an amalgamated subgroup is locally infinite.* (5·5)

(The reader is reminded that a group is called *locally infinite* if it has no elements $\neq 1$ of finite order.)

We remark that this is no longer generally valid for the generalized free product with different amalgamated subgroups. This is shown by the following example:

EXAMPLE. Let P be generated by three elements a, b, c with the defining relations

$$a^{-1}ba = b^{-1}, \quad b^{-1}cb = c^{-1}, \quad c^{-1}ac = a^{-1}. \tag{5·6}$$

We denote the subgroups generated by pairs of these generators by

$$G_1 = \{a, b\}, \quad G_2 = \{b, c\}, \quad G_3 = \{c, a\},$$

and we first show that P is the generalized free product of G_1, G_2, G_3. It suffices to show that the relation

$$a^{-1}ba = b^{-1} \tag{5·7}$$

defines the group G_1; the other two groups are analogous. If in G_1 there were still other relations, which follow from (5·6) but not from (5·7) alone, then a or b would have to have finite order; for every element of G_1 can, on account of (5·7), be written in the form $a^\alpha b^\beta$, thus a hypothetical further relation in the form $a^\alpha b^\beta = 1$. If here $\beta = 0$, then a has finite order; if $\beta \neq 0$, then b^β permutes with a, hence $b^{2\beta} = 1$ and b has finite order. However, no such relation can hold in P; for if we add to the relations of P the further relation $c = 1$, then we are left with the infinite dihedral group with generators a and b and relations

$$a^2 = 1, \quad a^{-1}ba = b^{-1},$$

and in this b has infinite order. Thus b has infinite order also in P, and so have a and c because of the evident symmetry of (5·6). It follows that P is the generalized free product of G_1, G_2, G_3; and incidentally we see that G_1, G_2, G_3 are locally infinite.

P, however, is not locally infinite. We consider the element abc in P. This is not the unit element, for otherwise we would have.

$$ab = c^{-1} = b^{-1}cb = b^{-1}.b^{-1}a^{-1}.b = a^{-1}b^3,$$

hence $a^2 = b^2$ and $b^4 = 1$, contrary to what we have just proved. On the other hand,

$$(abc)^2 = abcabc = aba^{-1}cbc = b^{-1}cbc = c^{-1}c = 1;$$

hence P contains elements of order 2.

Chapter II. Existence criteria for generalized free products

6. *The compatibility conditions*

We now turn to the question under what conditions given groups with prescribed intersections can be fitted together to a generalized free product. We cannot, however, treat this question at all exhaustively. We begin by formulating it more precisely.

Let groups G_α be given, where α runs over a suitable non-empty index set A. In every G_α and to every index $\beta \in$ A let a subgroup $H_{\alpha\beta}$ be distinguished; $H_{\alpha\alpha}$ is always to be the whole group† G_α. We then ask: does there exist a group P which is the generalized free product of groups G'_α with intersections

$$H'_{\alpha\beta} = G'_\alpha \cap G'_\beta = H'_{\beta\alpha},$$

where G'_α is to be isomorphic to G_α and $H'_{\alpha\beta}$ is to correspond to $H_{\alpha\beta}$ (all $\alpha, \beta \in$ A)? In other words, there are to be isomorphic mappings ϕ_α of G_α on to G'_α,

$$G'_\alpha = G_\alpha \phi_\alpha,$$

such that always

$$H'_{\alpha\beta} = H_{\alpha\beta}\phi_\alpha.$$

If there is such a group P, we say that the 'generalized free product of the G_α with amalgamated $H_{\alpha\beta}$' (or simply the 'generalized free product of the G_α') exists.‡

Certain conditions necessary for the existence of the generalized free product of the G_α are evident. As $H_{\alpha\beta}$ and $H_{\beta\alpha}$ are to be mapped isomorphically on to the same group

$$G'_\alpha \cap G'_\beta = H'_{\alpha\beta} = H'_{\beta\alpha},$$

they have to be isomorphic themselves; the mapping

$$\iota_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1}$$

† $H_{\alpha\alpha}$ is not really needed, and has only been introduced for the sake of a slight simplification in expression.

‡ This is a somewhat inaccurate mode of expression, as the generalized free product is not constructed from the given groups themselves, but from isomorphic copies of these groups, and also because not only the amalgamated subgroups, but also the amalgamating isomorphisms enter the construction—it is, however, convenient and not ambiguous.

has to be an isomorphism of $H_{\alpha\beta}$ on to $H_{\beta\alpha}$, and

$$\iota_{\beta\alpha} = \phi_\beta \phi_\alpha^{-1}$$

is the inverse isomorphism of $\iota_{\alpha\beta}$. Moreover, the three intersections

$$H_{\alpha\beta} \cap H_{\alpha\gamma}, \quad H_{\beta\alpha} \cap H_{\beta\gamma}, \quad H_{\gamma\alpha} \cap H_{\gamma\beta}$$

must be mutually isomorphic, as they are to be mapped on to one and the same subgroup

$$G'_\alpha \cap G'_\beta \cap G'_\gamma$$

of P. Specifically, if $h_\alpha \in H_{\alpha\beta} \cap H_{\alpha\gamma}$, then $h_\alpha \iota_{\alpha\beta}$ must belong to $H_{\beta\gamma}$, hence to $H_{\beta\alpha} \cap H_{\beta\gamma}$, and

$$h_\alpha \iota_{\alpha\beta} \iota_{\beta\gamma} = h_\alpha \iota_{\alpha\gamma}.$$

In the same way one can formulate further necessary conditions which stem from the fact that certain subgroups of G_α, G_β, etc., which are intersections of groups $H_{\alpha\beta}, H_{\alpha\gamma}, \ldots$, are to be mapped on to one and the same intersection of groups $G'_\alpha, G'_\beta, \ldots$ in P. These further conditions are, however, all satisfied as soon as the stated conditions are assumed; this is shown by the following lemma and corollary (Hanna Neumann 1948).

LEMMA. *Let groups G_α (all suffixes range over an index set A) with subgroups $H_{\alpha\beta}$ and mappings $\iota_{\alpha\beta}$ be given with the following properties*:

I. *$\iota_{\alpha\beta}$ is an isomorphism of $H_{\alpha\beta}$ on to $H_{\beta\alpha}$;*

II. *$\iota_{\beta\alpha}$ is the inverse of $\iota_{\alpha\beta}$;*

III. *$\iota_{\alpha\beta}$ maps $H_{\alpha\beta} \cap H_{\alpha\gamma}$ (isomorphically) on to $H_{\beta\alpha} \cap H_{\beta\gamma}$.*

Let Γ be a subset of A which contains α and β. Then the intersections

$$D_\alpha = \bigcap_{\gamma \in \Gamma} H_{\alpha\gamma}, \quad D_\beta = \bigcap_{\gamma\in\Gamma} H_{\beta\gamma}$$

are isomorphic, and $\iota_{\alpha\beta}$ maps D_α on to D_β. (6·1)

Proof. It clearly suffices to show that

$$D_\alpha \iota_{\alpha\beta} \subseteq D_\beta.$$

If $d \in D_\alpha$ then $d \in H_{\alpha\beta}$, hence $d\iota_{\alpha\beta}$ is defined and $d\iota_{\alpha\beta} \in H_{\beta\alpha}$. Moreover, for every $\gamma \in \Gamma$ we have

$$d \in H_{\alpha\beta} \cap H_{\alpha\gamma},$$

hence
$$d\iota_{\alpha\beta} \in H_{\beta\alpha} \cap H_{\beta\gamma};$$

thus
$$d\iota_{\alpha\beta} \in \bigcap_{\gamma\in\Gamma} H_{\beta\gamma} = D_\beta,$$

and the lemma follows.

COROLLARY. *If with the same assumption the following further condition is satisfied*:

IV. *The mapping $\iota_{\alpha\beta}\iota_{\beta\gamma}$ coincides with $\iota_{\alpha\gamma}$ where it is defined (namely, in $H_{\alpha\beta} \cap H_{\alpha\gamma}$), then we have more generally*

$$\iota_{\alpha\beta}\iota_{\beta\gamma}\iota_{\gamma\delta}\ldots\iota_{\kappa\lambda} \quad \textit{and} \quad \iota_{\alpha\omega}\iota_{\omega\psi}\iota_{\psi\chi}\ldots\iota_{\mu\lambda}$$

coincide for those elements for which they are both defined. (6·2)

For it is easy to see that for such elements both mappings coincide with $\iota_{\alpha\lambda}$.

Conditions I to IV will sometimes be referred to as the *compatibility conditions*; they are clearly necessary for the existence of the generalized free product of the G_α, and we shall therefore only deal with systems of groups G_α, subgroups $H_{\alpha\beta}$, and mappings $\iota_{\alpha\beta}$ which satisfy these conditions.

7. *The canonic group*

Let $G_\alpha, H_{\alpha\beta}, \iota_{\alpha\beta}$ $(\alpha, \beta \in \mathrm{A})$ form such a system which satisfies the compatibility conditions I to IV. Let each group G_α be given by a system of generators $g_{\alpha\gamma}$ $(\gamma \in \Gamma_\alpha)$ with defining relations

$$r_{\alpha\rho}(\ldots, g_{\alpha\gamma}, \ldots) = 1 \quad (\rho \in \mathrm{P}_\alpha). \tag{7·1}$$

Then we consider the group P', called the *canonic group*, which is obtained as follows: P' is generated by elements $g'_{\alpha\gamma}$ $(\alpha \in \mathrm{A}, \gamma \in \Gamma_\alpha)$ which correspond to the generators of all the groups G_α; the defining relations of P' are to be

$$r_{\alpha\rho}(\ldots, g'_{\alpha\gamma}, \ldots) = 1 \quad (\alpha \in \mathrm{A},\ \rho \in \mathrm{P}_\alpha), \tag{7·1'}$$

and, moreover, to every pair of corresponding elements

$$h_{\alpha\beta} = h_{\alpha\beta}(\ldots, g_{\alpha\gamma}, \ldots) \in H_{\alpha\beta}$$

and

$$h_{\alpha\beta}\iota_{\alpha\beta} = h_{\beta\alpha} = h_{\beta\alpha}(\ldots, g_{\beta\gamma}, \ldots) \in H_{\beta\alpha},$$

the ('amalgamating') relation

$$h_{\alpha\beta}(\ldots, g'_{\alpha\gamma}, \ldots) = h_{\beta\alpha}(\ldots, g'_{\beta\gamma}, \ldots). \tag{7·2'}$$

Throughout the present chapter the canonic group will be denoted by P'; we now proceed to study this group more closely. We shall see in the course of this chapter that P' is isomorphic to the generalized free product of the G_α with amalgamated $H_{\alpha\beta}$ if this product exists at all.

The correspondence

$$g_{\alpha\gamma} \to g'_{\alpha\gamma}$$

(for fixed $\alpha \in \mathrm{A}$ and all $\gamma \in \Gamma_\alpha$) generates a homomorphism θ_α of G_α on to a certain subgroup $G'_\alpha = G_\alpha\theta_\alpha$ of P'; for the elements $g'_{\alpha\gamma}$ satisfy at least the same relations as the generators $g_{\alpha\gamma}$ of G_α. The subgroups G'_α clearly generate P'. Moreover, we have, for every pair of corresponding elements $h \in H_{\alpha\beta}$, $h\iota_{\alpha\beta} \in H_{\beta\alpha}$,

$$h\theta_\alpha = h\iota_{\alpha\beta}\theta_\beta; \tag{7·3'}$$

hence corresponding subgroups $H_{\alpha\beta}\theta_\alpha$ and $H_{\beta\alpha}\theta_\beta$ coincide in P', and elements which correspond under the given isomorphisms $\iota_{\alpha\beta}$ are mapped on to one and the same element of P'. In P' we then have evidently for all $\alpha, \beta \in \mathrm{A}$,

$$H_{\alpha\beta}\theta_\alpha = H_{\beta\alpha}\theta_\beta \subseteq G'_\alpha \cap G'_\beta.$$

We now show, in analogy to theorem (1·1) (p. 505), that P' is in a certain sense (cf. Bates 1947) the *freest* group of its kind.

LEMMA. *Let P'' be a group which contains to every G_α a homomorphic map*

$$G''_\alpha = G_\alpha\eta_\alpha$$

in such a way that elements which correspond under the given isomorphisms $\iota_{\alpha\beta}$ are mapped on to one and the same element of P'', i.e. more precisely: such that for every $h \in H_{\alpha\beta}$ $(\alpha,\ \beta \in \mathrm{A})$ we have

$$h\eta_\alpha = h\iota_{\alpha\beta}\eta_\beta; \tag{7·3''}$$

then there exists a homomorphism ϕ of P' on to the subgroup of P'' generated by the G''_α; ϕ maps each G'_α $(\alpha \in \mathrm{A})$ on to G''_α, and for every $g \in G_\alpha$ it satisfies

$$g\theta_\alpha\phi = g\eta_\alpha. \tag{7·4}$$

Proof. As η_α is a homomorphism of G_α on to G''_α, the elements

$$g''_{\alpha\gamma} = g_{\alpha\gamma}\eta_\alpha$$

satisfy at least the same relations as their originals, that is,

$$r_{\alpha\rho}(\ldots, g''_{\alpha\gamma}, \ldots) = 1 \quad (\rho \in \mathrm{P}_\alpha). \tag{7·1″}$$

Moreover, we have, by assumption (7·3″), for every pair of corresponding elements

$$h_{\alpha\beta} = h_{\alpha\beta}(\ldots, g_{\alpha\gamma}, \ldots) \in H_{\alpha\beta}$$

and
$$h_{\alpha\beta}\iota_{\alpha\beta} = h_{\beta\alpha} = h_{\beta\alpha}(\ldots, g_{\beta\gamma}, \ldots) \in H_{\beta\alpha}$$

the amalgamating relation

$$h_{\alpha\beta}(\ldots, g''_{\alpha\gamma}, \ldots) = h_{\beta\alpha}(\ldots, g''_{\beta\gamma}, \ldots). \tag{7·2″}$$

The elements $g''_{\alpha\gamma} (\alpha \in \mathrm{A}, \gamma \in \Gamma_\alpha)$ therefore satisfy at least the same relations as the generators $g'_{\alpha\gamma}$ of P'; hence the mapping

$$g'_{\alpha\gamma} \to g''_{\alpha\gamma}$$

generates an homomorphism ϕ of P' on to the subgroup of P'' generated by the $g''_{\alpha\gamma}$; here G'_α is evidently mapped on to G''_α; more exactly: for every $g \in G_\alpha$ the element $g\theta_\alpha$ is mapped on to $g\eta_\alpha$, and the lemma follows.

LEMMA. *P' is the generalized free product of the groups G'_α.* (7·5)

Proof. We choose as systems $\mathfrak{E}_\alpha$ of generators of G'_α simply all the elements of G'_α; their union

$$\mathfrak{E} = \bigcup_{\alpha \in \mathrm{A}} \mathfrak{E}_\alpha$$

then evidently generates P'. As system $\mathfrak{R}_\alpha$ of defining relations of G'_α we simply have all the equations

$$abc^{-1} = 1$$

that express the fact that the product of the two elements a, b in G'_α is just the element $c \in G'_\alpha$. We wish to show that the union of all these systems of relations,

$$\mathfrak{R} = \bigcup_{\alpha \in \mathrm{A}} \mathfrak{R}_\alpha,$$

suffices to define P'. As G'_α is a homomorphic image of G_α, the relations $\mathfrak{R}_\alpha$ must at least entail the relations

$$r_{\alpha\rho}(\ldots, g'_{\alpha\gamma}, \ldots) = 1 \quad (\rho \in \mathrm{P}_\alpha), \tag{7·1′}$$

which correspond to the defining relations of G_α. If, moreover, we are given a pair of corresponding elements

$$h_{\alpha\beta} = h_{\alpha\beta}(\ldots, g_{\alpha\gamma}, \ldots) \in H_{\alpha\beta}$$

and
$$h_{\alpha\beta}\iota_{\alpha\beta} = h_{\beta\alpha} = h_{\beta\alpha}(\ldots, g_{\beta\gamma}, \ldots) \in H_{\beta\alpha},$$

then $\mathfrak{R}_\alpha$ entails for the element h' which corresponds to $h_{\alpha\beta}$ in G'_α

$$h' = h_{\alpha\beta}(\ldots, g'_{\alpha\gamma}, \ldots);$$

but the same element h' corresponds also to the element $h_{\beta\alpha} \in G'_\beta$, because of (7·3′); it follows then from $\mathfrak{R}_\beta$ that also

$$h' = h_{\beta\alpha}(\ldots, g'_{\beta\gamma}, \ldots).$$

Hence also
$$h_{\alpha\beta}(\ldots, g'_{\alpha\gamma}, \ldots) = h_{\beta\alpha}(\ldots, g'_{\beta\gamma}, \ldots) \tag{7·2′}$$

follows from the system $\mathfrak{R}$ of relations. Thus all the relations which we have used in the definition of P' follow from the system $\mathfrak{R}$, and the lemma follows.

8. *Existence criteria*

We are now in a position to establish the following criterion:

THEOREM. *The generalized free product of the groups G_α with amalgamated subgroups $H_{\alpha\beta}$ exists if, and only if,*

(1) *all the mappings θ_α of G_α on to G'_α are isomorphisms, and*

(2) *they map the subgroups $H_{\alpha\beta}$ on to the corresponding intersections $G'_\alpha \cap G'_\beta$.* (8·1)

Proof. If these two conditions are satisfied, then P' itself is the required generalized free product. Conversely, assume the existence of the generalized free product P of the G_α with amalgamated subgroups $H_{\alpha\beta}$. Let P be generated by subgroups $G''_\alpha = G_\alpha \eta_\alpha$, where the η_α are isomorphisms† and where always

$$H_{\alpha\beta}\eta_\alpha = H_{\beta\alpha}\eta_\beta = G''_\alpha \cap G''_\beta.$$

(More exactly, corresponding elements $h \in H_{\alpha\beta}$ and $h\iota_{\alpha\beta} \in H_{\beta\alpha}$ are to have the same map

$$h\eta_\alpha = h\iota_{\alpha\beta}\eta_\beta.)$$

Thus the mapping
$$\phi_\alpha = \eta_\alpha^{-1}\theta_\alpha$$
maps G''_α homomorphically on to G'_α in such a way that every element $h'' \in G''_\alpha \cap G''_\beta$ has the same map under both ϕ_α and ϕ_β:

$$h''\phi_\alpha = h''\phi_\beta \in G'_\alpha \cap G'_\beta.$$

Hence we can apply theorem (1·1) (p. 505) and extend all ϕ_α simultaneously to a homomorphism ϕ of P on to P'. On the other hand, there also exists, by lemma (7·4), a homomorphism η which maps P' on to P in such a way that for every $g \in G_\alpha$

$$g\theta_\alpha\eta = g\eta_\alpha.$$

But then $\phi\eta$ is an endomorphism of P which maps each G''_α identically on to itself; and as the G''_α together generate P, then $\phi\eta$ is the identical automorphism of P. In the same manner one shows that $\eta\phi$ is the identical automorphism of P'. Thus η and ϕ are mutually inverse isomorphisms, and P and P' are isomorphic. But here G'_α and G''_α just correspond according to $\phi_\alpha = \eta_\alpha^{-1}\theta_\alpha$, whence $\theta_\alpha = \eta_\alpha\phi_\alpha$ must themselves be isomorphisms. Moreover, $H_{\alpha\beta}$ gets mapped just on to the intersection $G'_\alpha \cap G'_\beta$, which is isomorphic to $G''_\alpha \cap G''_\beta$. This proves that the conditions are also necessary, and the theorem follows.

The two conditions (1) and (2) of theorem (8·1) each give rise to a further criterion.

THEOREM. *The mappings θ_α of G_α on to G'_α are all isomorphic if, and only if, there is a group P'' and to every $\alpha \in \mathrm{A}$ an isomorphism η_α* of *G_α on to a subgroup*

$$G''_\alpha = G_\alpha\eta_\alpha$$

of P'' such that corresponding elements $h \in H_{\alpha\beta}$ and $h\iota_{\alpha\beta} \in H_{\beta\alpha}$ have the same map

$$h\eta_\alpha = h\iota_{\alpha\beta}\eta_\beta. \tag{8·21}$$

(For the proof, see p. 521). We then have evidently

$$H_{\alpha\beta}\eta_\alpha \subseteq G''_\alpha \cap G''_\beta,$$

but the left-hand side can be properly contained in the right-hand side. If P'' satisfies the conditions of this theorem, we say it has the *isomorphism property*.

† Thus we could identify G''_α with G_α and think of P as composed of the given groups G_α themselves.

THEOREM. *The necessary and sufficient condition for $g \in G_\alpha$ and $g\theta_\alpha \in G'_\beta$ to imply $g \in H_{\alpha\beta}$ (which entails in particular*

$$H_{\alpha\beta}\theta_\alpha = G'_\alpha \cap G'_\beta)$$

is that there is a group P'' and to every $\alpha \in \mathrm{A}$ a homomorphism η_α of G_α on to a subgroup

$$G''_\alpha = G_\alpha \eta_\alpha$$

of P'' such that corresponding elements $h \in H_{\alpha\beta}$ and $h\iota_{\alpha\beta} \in H_{\beta\alpha}$ have the same map

$$h\eta_\alpha = h\iota_{\alpha\beta}\eta_\beta$$

and that $g \in G_\alpha$ and $g\eta_\alpha \in G''_\beta$ always implies $g \in H_{\alpha\beta}$. (8·22)

We then also have evidently $$H_{\alpha\beta}\eta_\alpha = G''_\alpha \cap G''_\beta. \tag{8·3}$$

Here the η_α need not be isomorphisms. If (8·3) is satisfied, we say P'' has the *weak intersection property*. But the condition entering theorem (8·22), namely, that only elements of $H_{\alpha\beta}$ are mapped into the intersection $G''_\alpha \cap G''_\beta$, is a rather more powerful restriction; if it is satisfied, we say P'' has the *strong intersection property*. Theorem (8·22) means that if P'' has the strong intersection property then so has P'; this is not true, however, of the weak intersection property, as, for example, the trivial group $P'' = \{1\}$ always trivially has this property. We now turn to the proof of the two theorems.

Proof of theorems (8·21) *and* (8·22). The necessity of the stated conditions follows immediately in both cases if one puts $P'' = P$ and $\eta_\alpha = \theta_\alpha$. We now assume conversely that there exist a group P'' and mappings η_α with the respective properties; thus P'' is to have either the isomorphism property or the strong intersection property. By lemma (7·4) there is a homomorphism ϕ of P' on to the subgroup of P'' generated by the G''_α which maps each G'_α on to G''_α such that

$$g\theta_\alpha\phi = g\eta_\alpha.$$

If the η_α are isomorphisms, they are one-to-one; then the θ_α must also be one-to-one, hence isomorphisms, and theorem (8·21) follows. On the other hand, if $g \in G_\alpha$ and $g\eta_\alpha \in G''_\beta$ always implies $g \in H_{\alpha\beta}$, and if we have $g \in G_\alpha$ and $g\theta_\alpha \in G'_\beta$, then

$$g\theta_\alpha\phi = g\eta_\alpha \in G'_\beta\phi = G''_\beta,$$

hence $g \in H_{\alpha\beta}$; thus also P' has the strong intersection property, and theorem (8·22) follows.

One can expresss the contents of theorems (8·21) and (8·22) simply as follows: If there is any group which has the isomorphism property or the strong intersection property, then P' has the same property. We now combine the criteria for the existence of the generalized free product to the following theorem.

THEOREM (Hanna Neumann 1948). *Necessary and sufficient condition for the existence of the generalized free product of the groups G_α with the subgroups $H_{\alpha\beta}$ amalgamated according to the isomorphisms $\iota_{\alpha\beta}$ is that there is a group P'' and to every $\alpha \in \mathrm{A}$ an isomorphism η_α of G_α on to a subgroup $G''_\alpha = G_\alpha\eta_\alpha$ of P'', such that corresponding elements $h \in H_{\alpha\beta}$ and $h\iota_{\alpha\beta} \in H_{\beta\alpha}$ have the same map*

$$h\eta_\alpha = h\iota_{\alpha\beta}\eta_\beta,$$

and, moreover, always $$H_{\alpha\beta}\eta_\alpha = G''_\alpha \cap G''_\beta. \tag{8·3}$$

Briefly: the generalized free product in question exists if, and only if, there is a group P'' which has the isomorphism property as well as the weak intersection property. (8·4)

Proof. The necessity of the stated condition for the existence of the generalized free product is seen from the fact that this has itself the isomorphism and intersection properties. The sufficiency follows from theorems (8·1), (8·21) and (8·22); we only have to show that theorem (8·22) applies, that is to say, that the weak intersection property (8·3) of P'' together with the isomorphism property implies the strong intersection property. Let then $g \in G_\alpha$ and $g\eta_\alpha \in G''_\beta$; then

$$g\eta_\alpha \in G''_\alpha \cap G''_\beta = H_{\alpha\beta}\eta_\alpha.$$

As η_α is an isomorphism, $g\eta_\alpha$ has only a single original g, and this then lies in $H_{\alpha\beta}$. Thus P'' has the strong intersection property, and the theorem follows.

9. *Amalgams and the canonic homomorphism*

The theory here developed can be reformulated in terms of amalgams of groups (Baer 1949; Hanna Neumann 1950). A brief sketch may suffice to give an idea of this approach.†

The *amalgam* A of the groups G_α (where again α ranges over some index set A) is an 'incomplete group' consisting of the elements of the groups G_α, with the product of two elements of A defined if, and only if, they both lie in (at least) one and the same group G_α. If two elements belong to more than one of the groups G_α, then their product is to be the same in all the groups in which they lie, so that in the amalgam their product is uniquely defined. The groups G_α are called the *constituent groups* of the amalgam. The intersection of two constituent groups is a group

$$G_\alpha \cap G_\beta = H_{\alpha\beta} = H_{\beta\alpha},$$

which may consist of the unit element (common to all groups G_α) alone. The amalgam of given groups can be formed so that given subgroups are identified with their intersections, according to given isomorphisms between them, precisely when the groups, subgroups, isomorphisms satisfy the compatibility conditions I to IV (cf. §6).

A *homomorphism* of the amalgam A into a group P is a mapping η of A into P such that if a, b are two elements of A whose product is defined in A, then

$$(ab)\,\eta = a\eta\, b\eta$$

in P. If there exists a *one-to-one homomorphism*‡ of A into a group P, we say the amalgam is *embeddable* in the group.

To decide whether A is embeddable in a group, we associate with A a particular group P' and a particular homomorphism θ of A into the group: P' is generated by elements

$$a' = a\theta$$

corresponding to the elements a of A; the defining relations of P' are

$$a'b' = c'$$

whenever $$a' = a\theta, \quad b' = b\theta, \quad c' = c\theta \quad \text{and} \quad ab = c$$

in A. We call P' and θ the *canonic group* and the *canonic homomorphism* of A respectively. The canonic group is easily recognized as the group denoted equally by P' earlier in this chapter; the canonic homomorphism θ combines within itself all the homomorphisms θ_α.

† This approach was used in Neumann & Neumann (1953), and I am taking the liberty of some verbatim quotation (without indication) from that paper.

‡ We reserve the term *isomorphism* for a one-to-one homomorphism whose inverse mapping is also a homomorphism.

THEOREM. *If η is a homomorphism of the amalgam A into a group P, then there is a homomorphism ϕ of the canonic group P' of A into P such that*

$$\eta = \theta\phi.$$

If η is one-to-one, then θ must be one-to-one. The amalgam is embeddable in a group if, and only if, the canonic homomorphism θ is one-to-one. (9·1)

The first part of the theorem is the counterpart of lemma (7·4); the second part is an immediate consequence of the factorization of η, for if a product of mappings is one-to-one, then the left-hand factor must be one-to-one; and the last part is then obvious. If the canonic homomorphism is one-to-one, then we say that P' is 'freely generated by A' or 'the generalized free product of the amalgam'. Lemma (7·5) shows that P' is in any case the generalized free product of the amalgam

$$A' = A\theta.$$

It is sometimes convenient to have a representation of the canonic group P' by fewer generators and relations than were used for its definition.

THEOREM. *Let $H_{\alpha\beta}$ be generated by elements $h_{\alpha\beta\gamma}$ $(\gamma \in \Gamma_{\alpha\beta})$ with defining relations*

$$r_\rho(\ldots, h_{\alpha\beta\gamma}, \ldots) = 1 \quad (\rho \in \mathrm{P}_{\alpha\beta}); \tag{9·3}$$

let G_α be generated by the $h_{\alpha\beta\gamma}$ $(\beta \in \mathrm{A}, \gamma \in \Gamma_{\alpha\beta})$ and further elements $g_{\alpha\delta}$ $(\delta \in \Delta_\alpha)$ with defining relations (9·3) *and*

$$s_\sigma(\ldots, g_{\alpha\delta}, \ldots, h_{\alpha\beta\gamma}, \ldots) = 1 \quad (\sigma \in \Sigma_\alpha). \tag{9·4}$$

Then the elements $\quad h'_{\alpha\beta\gamma} = h_{\alpha\beta\gamma}\theta, \quad g'_{\alpha\delta} = g_{\alpha\delta}\theta \quad (\alpha, \beta \in \mathrm{A},\ \gamma \in \Gamma_{\alpha\beta},\ \delta \in \Delta_\alpha)$

generate P', and the relations

$$r_\rho(\ldots, h'_{\alpha\beta\gamma}, \ldots) = 1, \tag{9·3'}$$

$$s_\sigma(\ldots, g'_{\alpha\delta}, \ldots, h'_{\alpha\beta\gamma}, \ldots) = 1 \tag{9·4'}$$

form a system of defining relations. (9·2)

The proof is omitted.

If η is a homomorphism of the amalgam A into a group P, then every constituent group G_α is mapped homomorphically on to a subgroup $G_\alpha\eta$ of P. The intersections $H_{\alpha\beta}$ are mapped into the intersections of the corresponding groups

$$H_{\alpha\beta}\eta \subseteq G_\alpha\eta \cap G_\beta\eta.$$

If η is one-to-one, then its restriction to G_α is also one-to-one, that is an isomorphism; moreover, then

$$H_{\alpha\beta}\eta = G_\alpha\eta \cap G_\beta\eta; \tag{9·5}$$

for otherwise two distinct elements of G_α, G_β, outside their intersection, would have the same map under η. Even more is true: no element of A outside $H_{\alpha\beta}$ is then mapped into $G_\alpha\eta \cap G_\beta\eta$, that is, $H_{\alpha\beta}$ is the complete inverse image of $G_\alpha\eta \cap G_\beta\eta$:

$$H_{\alpha\beta} = (G_\alpha\eta \cap G_\beta\eta)\eta^{-1}. \tag{9·6}$$

If (9·5) or (9·6) is satisfied for all $\alpha, \beta \in \mathrm{A}$, $\alpha \neq \beta$, then the homomorphism η (or, more loosely, the group P) has the *weak* or *strong intersection property* respectively. Clearly the strong implies the weak intersection property. If the restriction of η to every constituent G_α is an isomorphism, then η (or, more loosely, P) has the *isomorphism property*. We have just seen that if η is one-to-one, then it has all three properties. The converse is also true; this is seen from the first part of the following theorem, which summarizes theorems (8·4), (8·21) and (8·22):

THEOREM. *If η has the isomorphism property and the strong, or even only the weak, intersection property, then η is one-to-one. There is a homomorphism η with the isomorphism property if, and only if, the canonic homomorphism θ has the isomorphism property. There is a homomorphism η with the strong intersection property if, and only if, θ has the strong intersection property.* (9·7)

If there are two homomorphisms, one with the isomorphism property, the other with the strong intersection property, then θ has both properties and is therefore one-to-one, and the amalgam is then embeddable in a group. It suffices for this also that there is a single homomorphism having both the isomorphism and weak intersection properties. But it is not sufficient that there is one homomorphism with the isomorphism property and another with the weak intersection property; because the latter entails no restriction whatever on the amalgam: the trivial homomorphism on to the unit group clearly always has the weak intersection property.

10. *Embedding in an Abelian group*

It may be noted that in this chapter no really essential use has been made of the group property of the algebraic systems under investigation. In fact much of the theory can be carried over, with only trivial changes, to other and rather more general algebraic systems. We briefly sketch the special case of *Abelian groups*.

An amalgam A may be embeddable not merely in a group, but even in an Abelian group. This can clearly be the case only if the constituent groups G_α of A are all Abelian; if this is the case, we call the amalgam itself *Abelian*. To decide whether A is embeddable in an Abelian group, we associate with A a particular Abelian group P^+ together with a homomorphism θ^+ of A into P^+. This group is generated by elements

$$a^+ = a\theta^+$$

corresponding to the elements a of A; the defining relations of P^+ are

$$a^+b^+ = c^+$$

whenever $a^+ = a\theta^+$, $b^+ = b\theta^+$, $c^+ = c\theta^+$ and

$$ab = c$$

in A; and, moreover,

$$a^+b^+ = b^+a^+$$

for every pair of generators $a^+ = a\theta^+$, $b^+ = b\theta^+$ of P^+. It is not difficult to see that P^+ is isomorphic to the factor group of the canonic group P' with respect to its commutator group. One proves the following facts in complete analogy to theorem (9·1):

THEOREM. *If η is a homomorphism of the amalgam A into an Abelian group P, then there is a homomorphism ϕ of the group P^+ into P such that*

$$\eta = \theta^+\phi.$$

If η is one-to-one, then θ^+ must be one-to-one. The amalgam is embeddable in an Abelian group if, and only if, the homomorphism θ^+ is one-to-one. (10·1)

If θ^+ is one-to-one, then we say that P^+ is 'the generalized free sum (or free Abelian product) of the amalgam (or of the groups G_α with amalgamated $H_{\alpha\beta}$)'.

11. *Examples*

To conclude this chapter, two examples will show how the generalized free product can fail to exist; they also illustrate the two approaches here presented, the first being couched in the language of groups, subgroups and amalgamating isomorphisms, the second in the language of amalgams.

EXAMPLE 1. Let G_1 be the alternating group of degree 4, order 12; let G_2 be the dihedral group of order 8; let G_3 be the cyclic group of order 6. G_1 is generated by two elements a, b with defining relations

$$a^2 = b^3 = (ab)^3 = (ab^{-1}ab)^2 = 1;$$

a and $b^{-1}ab$ generate a four-group (the direct product of two cycles of order 2). The group G_2 is generated by two elements c, d with defining relations

$$c^2 = d^2 = (cd)^4 = 1;$$

c and $d^{-1}cd$ again generate a four-group. Finally, G_3 is generated by an element e with the defining relation

$$e^6 = 1.$$

We put

$$H_{12} = \{a, b^{-1}ab\}, \quad H_{21} = \{c, d^{-1}cd\},$$
$$H_{13} = \{b\}, \quad H_{31} = \{e^2\},$$
$$H_{23} = \{d\}, \quad H_{32} = \{e^3\}.$$

One easily confirms that the mappings

$$a\iota_{12} = c, \quad (b^{-1}ab)\,\iota_{12} = d^{-1}cd;$$
$$b\iota_{13} = e^2; \quad d\iota_{23} = e^3$$

generate isomorphisms. The remaining three isomorphisms are inverse to these. Moreover,

$$H_{12} \cap H_{13} = H_{21} \cap H_{23} = H_{31} \cap H_{32} = \{1\},$$

hence the compatibility conditions are satisfied. We now form P'; this is generated by elements a', b', c', d', e' with the defining relations

$$a'^2 = b'^3 = (a'b')^3 = (a'b'^{-1}a'b')^2 = 1;$$
$$c'^2 = d'^2 = (c'd')^4 = 1; \quad e'^6 = 1;$$
$$a' = c', \quad b'^{-1}a'b' = d'^{-1}c'd';$$
$$b' = e'^2; \quad d' = e'^3.$$

If we put $a' = c' = f$, $e' = g$, then $b' = g^2$, $d' = g^3$; and P' is also generated by f and g with the defining relations

$$f^2 = (fg^2)^3 = (fg^{-2}fg^2)^2 = (fg^3)^4 = g^6 = 1,$$
$$g^{-2}fg^2 = g^{-3}fg^3.$$

The last relation implies that f and g commute, and the rest then implies that $f = 1$. Thus P' is the cyclic group of order 6 generated by g. As P' contains no subgroup isomorphic to G_1 or G_2, it cannot have the isomorphism property; θ_1 and θ_2 must be proper homomorphisms. It is not difficult to verify that P' does not have the strong intersection property either, but does have the weak intersection property. The generalized free product of G_1, G_2, G_3 with the given amalgamations does not exist.

EXAMPLE 2. We form the amalgam A of four free Abelian groups of rank 3:

$$G_1 = \{a, b, c\}, \quad G_2 = \{a, c, d\},$$
$$G_3 = \{a, d, b\}, \quad G_4 = \{a, u, v\},$$

the defining relations (not listed) being the commutativity relations. The intersections of the pairs of constituent groups are to be

$$H_{12} = \{a, c\},$$
$$H_{13} = \{a, b\},$$
$$H_{14} = \{a, bc^{-1}, u;\ bc^{-1} = au\},$$
$$H_{23} = \{a, d\},$$
$$H_{24} = \{a, cd^{-1}, u^{-1}v;\ cd^{-1} = u^{-1}v\},$$
$$H_{34} = \{a, db^{-1}, v;\ db^{-1} = v^{-1}\};$$

the commutativity relations are again not listed, but only the additional defining relations. The intersections three at a time and the intersection of all four constituent groups are all the same group, namely, $\{a\}$.

The canonic group

$$P' = \{a', b', c', d', u', v';\ b'c'^{-1} = a'u', c'd'^{-1} = u'^{-1}v', d'b'^{-1} = v'^{-1}\}$$

can be generated also by a', b', c', d'; they are permutable in pairs, as each pair belongs to at least one constituent group, hence P' is Abelian, and can serve also as the group we denoted by P^+. Now in P'

$$1 = b'c'^{-1}.c'd'^{-1}.d'b'^{-1} = a'u'.u'^{-1}v'.v'^{-1} = a'.$$

Thus the canonic mapping θ maps the element $a \neq 1$ of the amalgam on to $a\theta = a' = 1$; hence θ is not one-to-one and the amalgam is not embeddable in a group. In fact θ acts as a proper homomorphism on each constituent group, that is, it lacks the isomorphism property. On the other hand, one easily verifies that P' is the free Abelian group generated by b', c', d'; and the groups which correspond to the G_i are

$$G'_1 = \{b', c'\}, \quad G'_2 = \{c', d'\}, \quad G'_3 = \{d', b'\},$$
$$G'_4 = \{u', v'\} = \{b'c'^{-1}, d'b'^{-1}\}.$$

They intersect in pairs in

$$G'_1 \cap G'_2 = \{c'\}, \qquad G'_1 \cap G'_3 = \{b'\},$$
$$G'_1 \cap G'_4 = \{b'c'^{-1}\}, \quad G'_2 \cap G'_3 = \{d'\},$$
$$G'_2 \cap G'_4 = \{c'd'^{-1}\}, \quad G'_3 \cap G'_4 = \{d'b'^{-1}\}.$$

One verifies at once that
$$(G'_\alpha \cap G'_\beta)\,\theta^{-1} = H_{\alpha\beta},$$

that is to say, θ has the strong (and thus also the weak) intersection property, and lacks only the isomorphism property.†

These examples have been chosen not only because they show how the isomorphism or intersection properties can fail, but also because they are in a certain sense minimal. It will be shown in the next chapter that the free product of *two* groups with an amalgamated

† This example was constructed by Hanna Neumann to disprove a conjecture of the author.

subgroup always exists; example 1 shows that already for three groups the generalized free product need no longer exist. It does, however, exist if at least two of the three groups are Abelian; this follows from a theorem of Hanna Neumann (1948, theorem 9·1); but it is not sufficient that only one of the groups is Abelian, as is again shown by the first example; and if one has four (or more) groups, then it does not even suffice that they are all Abelian, as the second example shows. By contrast the generalized free product of arbitrarily many cyclical or locally cyclical groups always exists (Hanna Neumann 1950). Further examples, to exhibit various reasons for which a generalized free product can fail to exist (or, equivalently, an amalgam can fail to be embeddable in a group), are to be found in Hanna Neumann (1948), Baer (1949), and Neumann & Neumann (1953); this last contains a systematic discussion of 'best possible' results in the direction here indicated.

Chapter III. Existence of the free product with one amalgamated subgroup

12. *The method of proof*

We now return to the important special case first studied by Schreier (1927), namely, that the intersections of the given groups all coincide to form a single group. In chapter I (§§2 *et seq.*) we investigated the properties of a group which was *given* as a free product with one amalgamated subgroup; in the present chapter we are concerned with the existence of such a free product.

We are then given groups G_α, where α again ranges over a suitable non-void index set A and in every G_α a subgroup H_α which is to play the role of all $H_{\alpha\beta}$ $(\beta\neq\alpha)$. The subgroups H_α are isomorphic to each other, and for every pair $\alpha,\beta\in$ A an isomorphism $\iota_{\alpha\beta}$ of H_α on to H_β is given; these isomorphisms satisfy

$$\iota_{\beta\alpha}=\iota_{\alpha\beta}^{-1},\quad \iota_{\alpha\beta}\iota_{\beta\gamma}=\iota_{\alpha\gamma}.$$

In this case the question of the existence of the generalized free product can be answered completely; the generalized free product always does exist.

The method of proof used by Schreier first constructs a normal form like that in chapter I, §2, for the elements of the product which is to be constructed, then defines a multiplication of these normal forms, and finally shows that they then form a group; this is just the required generalized free product. The most laborious part of this proof is the verification of the associative law of normal form multiplication. This difficulty is avoided by van der Waerden (1948), who introduces the product of the given groups as a permutation group of the set of normal words; this is our group Σ of chapter I (p. 509). In fact, van der Waerden deals only with the case of the ordinary free product (with only the trivial group amalgamated); but there appears to be no obstacle to an extension of his proof to the case here considered.

We here follow a similar way, which may perhaps not lead to the goal more rapidly than Schreier's or especially van der Waerden's, but which allows us to derive some new supplementary results. We also construct a permutation group—different, in general, from van der Waerden's—rather than the required generalized free product itself, and we then derive the existence of this latter by an application of theorem (8·4) (p. 521).

It is convenient to start off not with the groups G_α with their separate subgroups H_α and amalgamating isomorphisms $\iota_{\alpha\beta}$, but rather with the amalgam (cf. §9) of the groups; in other words, we think of the subgroups H_α as already amalgamated, according to the $\iota_{\alpha\beta}$, to a single subgroup H. Thus the amalgam, which we again denote by A, consists of the groups G_α; and every pair of different constituent groups has the same intersection:

$$G_\alpha \cap G_\beta = H \quad (\alpha \neq \beta,\ \alpha, \beta \in \mathrm{A}).$$

In A two elements have a product if, and only if, they belong to the same G_α; if they both lie in more than one G_α, then they lie in H (and thus in all constituent groups simultaneously), and their product is the same, whichever constituent group we may wish to assign them to.

13. *A permutation group and its properties*

As in chapter I we begin by choosing in every group G_α a system S_α of representatives of the left cosets modulo H. Then every element $g \in G_\alpha$ is uniquely represented in the form

$$g = sh \quad (s \in S_\alpha,\ h \in H).$$

We again assume, for the sake of simplicity, that H itself is always represented by the unit element, that is, $1 \in S_\alpha$ for all α.

We now consider the Cartesian product K of the given groups G_α, that is, the set of all functions $k = k(\alpha)$ of a variable ranging over the index set A and taking values in G_α:

$$k(\alpha) \in G_\alpha.$$

One may think of K as the direct product of all groups G—the unrestricted direct product† if there are infinitely many of them—but the multiplication of elements of K which is then defined is inessential for our purpose; we shall only be concerned with certain permutations of the elements of K.

With every element $g_\alpha \in G_\alpha$ we associate a mapping $\rho_\alpha(g_\alpha)$ of K into itself by putting, for $k \in K$,

$$k\rho_\alpha(g_\alpha) = k',$$

where $k'(\alpha) = k(\alpha) . g_\alpha$, $k'(\beta) = k(\beta)$ when $\beta \neq \alpha$. If K is interpreted as the direct product of the G_α, then this defines the right multiplication by the element whose component in G_α is g_α, in every other G_β the unit element. In any case it is obvious that $\rho_\alpha(g_\alpha)$ is a permutation of K, and that all these permutations $\rho_\alpha(g_\alpha)$, with g_α ranging over G_α, form a permutation group isomorphic to G_α; this group we denote by $\rho_\alpha(G_\alpha)$. Next we introduce a mapping $\pi_{\alpha\beta}$ of K into itself, for every pair $\alpha, \beta \in \mathrm{A}$, $\alpha \neq \beta$. If k is an arbitrary element of K, we first find the unique representations:

$$k(\alpha) = s_\alpha h_\alpha \quad (s_\alpha \in S_\alpha,\ h_\alpha \in H),$$
$$k(\beta) = s_\beta h_\beta \quad (s_\beta \in S_\beta,\ h_\beta \in H),$$

and then put

$$k\pi_{\alpha\beta} = k',$$

where

$$k'(\alpha) = s_\alpha h_\beta,$$
$$k'(\beta) = s_\beta h_\alpha,$$
$$k'(\gamma) = k(\gamma) \quad \text{for } \gamma \neq \alpha, \beta.$$

(Observe that $s_\alpha h_\beta$ is an element of G_α, and $s_\beta h_\alpha$ an element of G_β.) If $\pi_{\alpha\beta}$ is carried out twice in succession then one evidently obtains the identical permutation of K; hence $\pi_{\alpha\beta}$ is itself a permutation of K. One also easily sees that $\pi_{\alpha\beta}$ and $\pi_{\beta\alpha}$ are the same permutation.

† The restricted direct product would serve just as well.

Now let h be an element of H. We show

$$\pi_{\alpha\beta}\rho_\beta(h)\,\pi_{\alpha\beta} = \rho_\alpha(h); \tag{13·1}$$

that is to say, if K is thought of as the direct product, the 'right multiplication' defined by h *qua* element of G_β is transformed by $\pi_{\alpha\beta}$ into the 'right multiplication' defined by h *qua* element of G_α. Put again, for arbitrary $k \in K$,

$$\left.\begin{aligned} k(\alpha) &= s_\alpha h_\alpha \quad (s_\alpha \in S_\alpha,\ h_\alpha \in H),\\ k(\beta) &= s_\beta h_\beta \quad (s_\beta \in S_\beta,\ h_\beta \in H).\end{aligned}\right\} \tag{13·2}$$

Putting also $$k\pi_{\alpha\beta} = k_1, \quad k_1\rho_\beta(h) = k_2, \quad k_2\pi_{\alpha\beta} = k', \tag{13·3}$$

then (with γ always different from both α and β)

$$\begin{aligned} k_1(\alpha) &= s_\alpha h_\beta,\\ k_1(\beta) &= s_\beta h_\alpha,\\ k_1(\gamma) &= k(\gamma);\\ k_2(\alpha) &= k_1(\alpha) = s_\alpha h_\beta,\\ k_2(\beta) &= k_1(\beta)\,.\,h = s_\beta h_\alpha h,\\ k_2(\gamma) &= k_1(\gamma) = k(\gamma);\\ k'(\alpha) &= s_\alpha h_\alpha h = k(\alpha)\,.\,h,\\ k'(\beta) &= s_\beta h_\beta = k(\beta),\\ k'(\gamma) &= k_2(\gamma) = k(\gamma).\end{aligned}$$

Thus the effect of $\pi_{\alpha\beta}\rho_\beta(h)\,\pi_{\alpha\beta}$ on k is the same as that of $\rho_\alpha(h)$, and as k is an arbitrary element of K, (13·1) follows.

Next let two elements $g_\alpha \in G_\alpha$ and $g_\beta \in G_\beta$ be given. We show that

$$\pi_{\alpha\beta}\rho_\beta(g_\beta)\,\pi_{\alpha\beta} = \rho_\alpha(g_\alpha) \tag{13·4}$$

can hold *only if* g_β equals g_α, and thus $g_\beta = g_\alpha \in H$. For with the notation (13·2) and (13·3) we have again

$$\begin{aligned} k_1(\alpha) &= s_\alpha h_\beta,\\ k_1(\beta) &= s_\beta h_\alpha,\\ k_1(\gamma) &= k(\gamma),\end{aligned}$$

and

$$\begin{aligned} k_2(\alpha) &= k_1(\alpha) = s_\alpha h_\beta,\\ k_2(\beta) &= k_1(\beta)\,.\,g_\beta = s_\beta h_\alpha g_\beta,\\ k_2(\gamma) &= k_1(\gamma) = k(\gamma).\end{aligned}$$

If now $$s_\beta h_\alpha g_\beta = s'_\beta h' \quad (s'_\beta \in S_\beta,\ h' \in H), \tag{13·5}$$

then $$k_2(\beta) = s'_\beta h',$$

and

$$\begin{aligned} k'(\alpha) &= s_\alpha h',\\ k'(\beta) &= s'_\beta h_\beta,\\ k'(\gamma) &= k(\gamma).\end{aligned}$$

On the other hand, if $k'' = k\rho_\alpha(g_\alpha)$, we have

$$k''(\alpha) = s_\alpha h_\alpha g_\alpha,$$
$$k''(\beta) = k(\beta) = s_\beta h_\beta,$$
$$k''(\gamma) = k(\gamma).$$

In order that k' and k'' may be the same element of K, we must then have

$$s'_\beta = s_\beta, \quad h' = h_\alpha g_\alpha.$$

Then we obtain from (13·5) that $$g_\beta = h_\alpha^{-1} h' \in H$$

and $$h' = h_\alpha g_\beta,$$

and our assertion follows.

Now let γ again be different from both α and β, and let elements $g_\beta \in G_\beta$ and $g_\gamma \in G_\gamma$ be given. We next show that
$$\pi_{\alpha\beta}\rho_\beta(g_\beta)\,\pi_{\alpha\beta} = \pi_{\alpha\gamma}\rho_\gamma(g_\gamma)\,\pi_{\alpha\gamma} \tag{13·6}$$

can hold *only if* g_β and g_γ are the same element, which then must lie in H. Let again k be an arbitrary element of K, and put

$$k(\alpha) = s_\alpha h_\alpha \quad (s_\alpha \in S_\alpha,\ h_\alpha \in H),$$
$$k(\beta) = s_\beta h_\beta \quad (s_\beta \in S_\beta,\ h_\beta \in H),$$
$$k(\gamma) = s_\gamma h_\gamma \quad (s_\gamma \in S_\gamma,\ h_\gamma \in H).$$

If now, as before, $$s_\beta h_\alpha g_\beta = s'_\beta h' \quad (s'_\beta \in S_\beta,\ h' \in H), \tag{13·5}$$

and correspondingly $$s_\gamma h_\alpha g_\gamma = s''_\gamma h'' \quad (s''_\gamma \in S_\gamma,\ h'' \in H), \tag{13·7}$$

and if $$k\pi_{\alpha\beta}\rho_\beta(g_\beta)\,\pi_{\alpha\beta} = k',$$
$$k\pi_{\alpha\gamma}\rho_\gamma(g_\gamma)\,\pi_{\alpha\gamma} = k'',$$

then our earlier calculations show that

$$k'(\alpha) = s_\alpha h',$$
$$k'(\beta) = s'_\beta h_\beta,$$
$$k'(\gamma) = k(\gamma) = s_\gamma h_\gamma,$$

and $$k''(\alpha) = s_\alpha h'',$$
$$k''(\beta) = k(\beta) = s_\beta h_\beta,$$
$$k''(\gamma) = s''_\gamma h_\gamma.$$

For k' and k'' to be the same element of K we must then have

$$s'_\beta = s_\beta, \quad s''_\gamma = s_\gamma, \quad h' = h''.$$

Then (13·5) and (13·7) give $$g_\beta = h_\alpha^{-1} h' \in H,$$
$$g_\gamma = h_\alpha^{-1} h'' \in H,$$

and our assertion follows.

14. *Schreier's theorem*

We now single out one of the indices in A; let us denote it by 0. Let P'' be the group of permutations of K which is generated by $\rho_0(G_0)$ and all $\pi_{0\alpha}\rho_\alpha(G_\alpha)\,\pi_{0\alpha}$ $(\alpha \in \mathrm{A},\ \alpha \neq 0)$. If we introduce also π_{00} and put it equal to the identical permutation, then

$$P'' = \{\pi_{0\alpha}\rho_\alpha(G_\alpha)\,\pi_{0\alpha}\}_{\alpha \in \mathrm{A}}.$$

We can now prove:

Theorem. *The mapping η defined on the amalgam A by*

$$g_\alpha \eta = \pi_{0\alpha}\rho_\alpha(g_\alpha)\,\pi_{0\alpha} \quad (g_\alpha \in G_\alpha)$$

is one-valued on H; it is a one-to-one homomorphism of the amalgam into P''. (14·1)

Proof. If $h \in H$, then for all $\alpha \in \mathrm{A}$,

$$h\eta = \pi_{0\alpha}\rho_\alpha(h)\,\pi_{0\alpha} = \rho_0(h),$$

showing η to be one-valued on H. On each G_α the mapping η acts as an isomorphism, being obtained by first mapping G_α isomorphically on to $\rho_\alpha(G_\alpha)$, and then applying the inner automorphism induced by $\pi_{0\alpha}$. Thus η is a homomorphism of the amalgam A into P'' and has, moreover, the isomorphism property. It only remains to show that η also has the weak (and thus also the strong) intersection property. Putting

$$G''_\alpha = G_\alpha \eta = \pi_{0\alpha}\rho_\alpha(G_\alpha)\,\pi_{0\alpha},$$

let $$g'' \in G''_\alpha \cap G''_\beta \quad (\alpha \neq \beta,\ \alpha, \beta \in \mathrm{A}).$$

Then there are elements $g_\alpha \in G_\alpha$ and $g_\beta \in G_\beta$ such that

$$g_\alpha \eta = g_\beta \eta = g''.$$

This means that $$\pi_{0\alpha}\rho_\alpha(g_\alpha)\,\pi_{0\alpha} = \pi_{0\beta}\rho_\beta(g_\beta)\,\pi_{0\beta},$$

and we have seen that this (which is equation (13·6), p. 530, with $0, \alpha, \beta$ taking the place of α, β, γ) is possible only if $g_\alpha = g_\beta \in H$. Thus

$$g'' \in H\eta,$$

and $$H\eta \supseteq G_\alpha \eta \cap G_\beta \eta.$$

But also, trivially, $$H\eta \subseteq G_\alpha \eta \cap G_\beta \eta;$$

it follows that η has the weak intersection property and—as it has the isomorphism property—also the strong intersection property, and thus is one-to-one. This completes the proof of the theorem. Application of theorems (8·4) (p. 521) and (9·7) (p. 524) then gives the theorem of Schreier (1927):

Theorem. *The generalized free product of the amalgam of the groups G_α, with a single subgroup H amalgamated always exists. In particular the generalized free product of two groups always exists.* (14·2)

It may be remarked that our proof also establishes the existence of the ordinary free product, where only the trivial subgroups become amalgamated; but in this special case one can more simply apply theorem (8·4) to the *direct* product of the given factors, which certainly contains them isomorphically and with trivial intersections.

The use of what amounts to the regular permutation representation in this context is based on (independent) suggestions of Philip Hall, Graham Higman and Wielandt.

15. *Consequences of the proof and of Schreier's theorem*

The following is a corollary of our proof:

Corollary. *If the G_α are finitely many finite groups then the group P'' of theorem* (14·1) *is also finite.* (15·1)

For then K, as the Cartesian product of finitely many finite sets, is itself finite, and P'' is a permutation group of a finite set, hence also finite. One can easily give an upper bound for the order of P'' in terms of the orders of the G_α.

Corollary. *A finite amalgam of groups with one amalgamated subgroup is embeddable in a finite group.* (15·2)

The *proper* free product of groups with one amalgamated subgroup is, however, always infinite; such a product being 'proper' if it has at least two factors which contain the amalgamated subgroup properly.† To see this one only has to observe that in a proper free product with an amalgamated subgroup one can write down a normal word of length 2; this is necessarily cyclically reduced, hence represents an element of infinite order (cf. lemma (5·3), p. 514).

Corollary. *The* (*proper*) *free product of finitely many finite groups with an amalgamated subgroup is not simple; on the contrary it has a proper normal subgroup of finite index.* (15·3)

For, as we have just seen, the generalized free product is infinite; but P'' is finite by corollary (15·1), and a homomorphic image of the canonic group P' by lemma (7·4) (p. 518); and P' is isomorphic to the generalized free product because this exists (proof of theorem (8·1), p. 520).

Ruth Camm (1953) has constructed, by contrast, a group‡ which is the free product of two infinite groups with an amalgamated subgroup and which is, moreover, simple. Hence P'' is in certain circumstances isomorphic to the generalized free product, and the corollary cannot be extended to infinite groups.

We had at first distinguished carefully between the given groups G_α and their isomorphic copies which entered the definition of the generalized free product. But already in the notion of an amalgam, and in the phrase 'free product with *one* amalgamated subgroup' the distinction has become blurred; and we shall now use less care in this respect. Thus when the generalized free product of the G_α exists we shall think of it as composed of these groups themselves; and shall also generally identify isomorphic groups with each other where permissible and convenient.

The following result is an easy consequence of Schreier's theorem, and useful for some applications.

Theorem. *Let Q be a group and let K_α be certain subgroups of Q, where α ranges over an index set* A. *Let every group K_α be embeddable in a group G_α* (*in other words, there is an isomorphism of K_α on to a subgroup of G_α*). *Then Q and the G_α can be simultaneously embedded in a group P in such a way that* (*for all* $\alpha \in \mathrm{A}$)

$$G_\alpha \cap Q = K_\alpha$$

and (*for all* $\alpha, \beta \in \mathrm{A}$, $\alpha \neq \beta$)

$$G_\alpha \cap G_\beta = K_\alpha \cap K_\beta. \qquad (15·4)$$

† An amalgam is called 'proper' if it is not already a group, or if it contains two elements with no product in it.

‡ In fact continuously many different such groups.

Proof. We first form to every α the free product G^*_α of G_α and Q, amalgamating K_α *qua* subgroup of G_α with K_α *qua* subgroup of Q. Then we form the free product P of all G^*_α, amalgamating Q. The existence of the G^*_α and of P follows from theorem (14·2). Evidently P contains all G_α and Q as subgroups, namely, as subgroups of its factors G^*_α. In G^*_α, and hence also in P, the intersection of G_α and Q is

$$G_\alpha \cap Q = K_\alpha.$$

If $\alpha \neq \beta$, then
$$G_\alpha \cap G_\beta \subseteq G^*_\alpha \cap G^*_\beta = Q,$$
hence
$$G_\alpha \cap G_\beta \subseteq (G_\alpha \cap Q) \cap (G_\beta \cap Q) = K_\alpha \cap K_\beta.$$
But obviously
$$K_\alpha \cap K_\beta \subseteq G_\alpha \cap G_\beta;$$
hence finally
$$G_\alpha \cap G_\beta = K_\alpha \cap K_\beta,$$
and the theorem follows.

Corollary.† *Every group G can be embedded in a group G^* in which every element is, for every positive integer n, an nth power.* (15·5)

For obviously every cyclic group can be so embedded, either in the additive group R of all rational numbers, or in the multiplicative group S of all roots of unity. Now one embeds G according to theorem (15·4) in a group G_1 in which every cyclic subgroup of G is embedded in an isomorphic copy of R or S. Then every element of G is, for every positive integer n, the nth power of an element of G_1. Then one proceeds with G_1 as with G, that is, one embeds G_1 in a group G_2 in which every element of G_1 is an nth power, and so on. In this way one obtains an ascending sequence of groups

$$G \subseteq G_1 \subseteq G_2 \subseteq \ldots,$$

where every element of G_i is, for every positive n, an nth power in G_{i+1}. Finally, we take as G^* the union of this sequence:
$$G^* = \bigcup_i G_i.$$
Then G^* evidently has the required property.

We return once more to the question of the existence of the generalized free product of groups G_α with subgroups $H_{\alpha\beta}$ amalgamated (according to isomorphisms $\iota_{\alpha\beta}$), in other words, the question of the embeddability of the amalgam A of the groups G_α, where the intersections

$$G_\alpha \cap G_\beta = H_{\alpha\beta}$$

now need no longer all be one and the same group. If we denote by K_α the subgroup of G_α generated by all $H_{\alpha\beta}$ with fixed α, variable $\beta \neq \alpha$, then the K_α also form an amalgam B, say, with the same amalgamations:
$$K_\alpha \cap K_\beta = H_{\alpha\beta}.$$
For clearly
$$K_\alpha \cap K_\beta \subseteq G_\alpha \cap G_\beta = H_{\alpha\beta},$$
and also $H_{\alpha\beta} \subseteq K_\alpha$ and $H_{\alpha\beta} \subseteq K_\beta$, whence

$$K_\alpha \cap K_\beta \supseteq H_{\alpha\beta}.$$

We call B a *reduced amalgam*; by this we mean an amalgam in which each constituent is generated by its intersections with the other constituents. Now we can answer the question

† Cf. B. H. Neumann (1943*a*).

of the embeddability of the amalgam A provided we can answer it for the reduced amalgam B. This is achieved by the following 'reduction theorem' of Hanna Neumann†:

THEOREM. *The amalgam A of groups G_α with amalgamated $H_{\alpha\beta}$ can be embedded in a group if, and only if, its reduced subamalgam B (of the K_α with amalgamated $H_{\alpha\beta}$) can be so embedded.* (15·6)

Proof. Any embedding of the amalgam A in a group simultaneously embeds the reduced amalgam B. Assume conversely that B is embedded in a group Q. According to theorem (15·4) we can then embed Q in a group P which contains the groups G_α such that

$$G_\alpha \cap G_\beta = K_\alpha \cap K_\beta = H_{\alpha\beta};$$

in other words, P contains the amalgam A, and the theorem follows.

16. *The generalized direct product*

In analogy to the theory of the generalized free product one can also develop a theory of the generalized *direct* product; we shall give here a brief sketch only.

The group P is called the *generalized direct product*‡ of its subgroups $G_\alpha (\alpha \in \mathrm{A})$, if P is generated by the G_α, and if for every pair $\alpha \neq \beta$ in A every element of G_α permutes with every element of G_β. Denoting the intersection of two such groups again by

$$H_{\alpha\beta} = H_{\beta\alpha} = G_\alpha \cap G_\beta,$$

one sees that these $H_{\alpha\beta}$ must all be contained in the centre of P; for an element of $H_{\alpha\beta}$ is, *qua* element of G_α, permutable with all the elements of all the G_γ, $\gamma \neq \alpha$, and *qua* element of G_β it is also permutable with all the elements of G_α.

If, conversely, we are given an amalgam A of groups G_α $(\alpha \in \mathrm{A})$ with intersections $H_{\alpha\beta}$ and if, moreover, every $H_{\alpha\beta}$ lies in the centre of G_α (and, by symmetry, in the centre of G_β), then we can ask for the existence of the generalized direct product of the amalgam A, that is the generalized direct product of the groups G_α with the $H_{\alpha\beta}$ amalgamated. The following theorem gives a partial answer, in analogy to Schreier's theorem:

THEOREM. *If A is the amalgam of groups G_α with a single subgroup H amalgamated, where H lies in the centre of each constituent G_α, then the generalized direct product of A exists.* (16·1)

Proof. The proof is rather simpler than in the case of the generalized free product. But here the amalgam is not the best tool to use, and we return rather to the situation described at the beginning of chapter II (§6). This we do by providing ourselves with isomorphic copies G'_α of the given G_α, with isomorphisms ι_α, say, mapping G'_α on to G_α. The subgroup of G'_α which corresponds to H is denoted by H'_α. Thus $H'_\alpha \iota_\alpha = H$. Now we form the restricted direct product K of all G'_α. Then K contains in its centre the direct product H^* of the mutually isomorphic groups H'_α. An element $h^* \in H^*$ can be represented as a product

$$h^* = \Pi_{\alpha \in \mathrm{A}} h'_\alpha \quad (h'_\alpha \in H'_\alpha),$$

† Hanna Neumann (1948, theorem 5·0); for a simpler proof, cf. Baer (1949); the proof here presented is adapted from Neumann & Neumann (1950).

‡ This is the *restricted* generalized direct product (when A is infinite). We do not here introduce the unrestricted generalized direct product, nor yet the unrestricted free product. For the latter, cf. Graham Higman (1952).

where only a finite number of factors is different from 1. Those elements h^* for which

$$\Pi_{\alpha\in A} h'_\alpha \iota_\alpha = 1$$

form a subgroup N of H^*, which, as subgroup of the centre of K, is normal in K. The factor group $P = K/N$ is then the required direct product with amalgamated subgroups. To see this we remark that N has trivial intersection with each G'_α, hence G'_α—and thus also G_α—is isomorphically represented by the subgroup $G'_\alpha \cup N/N$ of K/N; these subgroups evidently generate P and are elementwise permutable. If $h_\alpha \in H'_\alpha$ and $h_\beta \in H'_\beta$ correspond to the same element $h \in H$, that is, if $h_\alpha \iota_\alpha = h_\beta \iota_\beta = h$, then $h_\alpha^{-1} h_\beta$ belongs to N, and h_α and h_β become the same element of P. On the other hand, one easily verifies that no element of G'_α outside H'_α is congruent to any element of any other group G'_β modulo N; hence the intersection of $G'_\alpha \cup N/N$ and $G'_\beta \cup N/N$ in P is exactly the subgroup corresponding to H; identifying $G'_\alpha \cup N/N$ with G_α we find that P is the direct product of the G_α with H amalgamated, and the theorem follows.

The existence of the generalized direct product with arbitrary amalgamated subgroups of the centre can be reduced, exactly as in the case of the generalized free product, to the existence of the generalized direct product of the reduced amalgam (Neumann & Neumann 1950). Here the subgroups K_α of G_α generated by the $H_{\alpha\beta}$ with α fixed are subgroups of the centre of G_α, thus Abelian. Hence the reduced amalgam is Abelian, and one is led again to investigate the possibility of embedding an Abelian amalgam in an Abelian group (cf. §10). For an Abelian amalgam the generalized direct product and the generalized free Abelian product (or generalized free sum) coincide. Thus example 2, §11 (p. 526), shows incidentally that the generalized direct product of four Abelian groups does not necessarily exist. The generalized direct product of *three* Abelian groups, however, always exists (Hanna Neumann 1951, theorem 9). If the generalized direct product of an Abelian amalgam exists then the generalized free product of the same amalgam also exists. The converse is true if the number of constituents is four, or less; but not in general when there are five constituents (Hanna Neumann 1951; Neumann & Neumann 1953).

Chapter IV. Application of the generalized free product to embedding problems

17. *Extension of an isomorphism of subgroups to an inner automorphism of a supergroup*

We now turn to some embedding problems, that is, the following type of question: does there exist a group with certain properties containing a given group, or several given groups? The question of the existence of the generalized free (or direct) product of given groups is itself a typical embedding problem; theorems (14·2), (15·4) and (16·1) and corollary (15·5) are embedding theorems. It is then not surprising that in dealing with embedding questions the free product with one amalgamated subgroup, whose existence is the contents of Schreier's theorem (theorem (14·2)), will present itself as the principal tool.

Let a group G be given with two subgroups A and B; we ask under what conditions G can be embedded in a supergroup† H in which A and B are conjugate. As conjugate subgroups

† This term translates the German 'Obergruppe', which is the convenient counterpart to 'Untergruppe' (subgroup).

are isomorphic, this certainly can only be possible if A and B are isomorphic. We shall now show that this trivially necessary condition is also sufficient.

THEOREM. (Higman, Neumann & Neumann 1949). *Let the subgroups A and B of G be isomorphic and let ϕ be a given isomorphism of A on to B. Then there is a group H containing G in which ϕ is induced by an inner automorphism; in other words, H contains an element t such that*

$$t^{-1}at = a\phi \qquad (17\cdot2)$$

for every $a \in A$. (17·1)

Proof. We form the free product of G with an infinite cyclic group,

$$K_1 = G * \{u\},$$

and the same again with another infinite cyclic group:

$$K_2 = G * \{v\}.$$

In K_1 we consider the subgroup L_1 generated by G and $u^{-1}Au$; this is the free product of G and $u^{-1}Au$:

$$L_1 = G * u^{-1}Au,$$

by an easy application of corollary (4·3) (p. 514) and theorem (4·1) (p. 512). The subgroup L_2 of K_2 generated by G and vBv^{-1} is likewise the free product of these groups:

$$L_2 = G * vBv^{-1}.$$

L_1 and L_2 are isomorphic; we obtain an isomorphism of L_1 on to L_2 if we map $G \subseteq L_1$ identically on to $G \subseteq L_2$, and every element $u^{-1}au \in u^{-1}Au$ on to the element $v(a\phi)\,v^{-1} \in vBv^{-1}$. We now form the free product H of K_1 and K_2, amalgamating L_1 and L_2 according to this isomorphism. H contains G as a subgroup; and for every $a \in A$ we have in H

$$u^{-1}au = v(a\phi)\,v^{-1}.$$

Putting $uv = t$, we see that

$$t^{-1}at = a\phi \qquad (17\cdot2)$$

for all elements of A, and the assertion follows.

COROLLARY. *Let elements a_α, b_α be given in the group G, where α ranges over a non-empty index set* A. *Necessary and sufficient condition for the system of simultaneous equations*

$$t^{-1}a_\alpha t = b_\alpha \quad (\alpha \in \mathrm{A})$$

to have a solution t in some supergroup H of G, is that the mapping $a_\alpha\phi = b_\alpha$ generates an isomorphism of the subgroup of G generated by the a_α on to the subgroup generated by the b_α. (17·3)

This is only a different, equivalent form of theorem (17·1).

COROLLARY. *Two elements a, b of a group G are conjugate in a suitable supergroup of G if, and only if, they have the same order.* (17·4)

Instead of the group generated by G, u and v constructed for the proof of the theorem, one can of course restrict consideration to the group generated by G and t.

COROLLARY. *The group generated by generators of G and the element t and defined by the defining relations of G and the further relations*

$$t^{-1}at = a\phi \qquad (17\cdot2)$$

(for all $a \in A$) contains G (isomorphically); t generates an infinite cycle which has only the unit element in common with G. (17·5)

That the relations (17·2) entail no new relations between elements of G follows immediately from theorem (17·1); if in the group defined in the corollary all elements of G are put equal to 1, then t only remains, and no relation for t; thus the rest of the corollary follows.

Corollary. *If G is denumerable, then H can be chosen denumerable.* (17·6)

Corollary. *If G is locally infinite, then H can be chosen locally infinite.* (17·7)

This follows from corollary (5·5) (p. 515) because H can be taken (as in the proof) as subgroup of a free product of locally infinite groups with one amalgamated subgroup.

18. *Philip Hall's alternative proof*

Another proof of theorem (17·1) is given, which uses permutation groups instead of free products with amalgamations, and which is due to Philip Hall (unpublished).

We consider the regular permutation representation of a group K by right multiplications; if $k^* \in K$, let $\rho(k^*)$ denote the corresponding right multiplication, that is, the mapping defined by

$$k\rho(k^*) = k \,.\, k^*$$

for all elements $k \in K$. If A is a subgroup of K, we denote by $\rho(A)$ the group generated by the right multiplications $\rho(a)$ (and in fact consisting of these right multiplications) which belong to the elements $a \in A$. This group is isomorphic to A.

Lemma. *If A and B are subgroups of K, then the corresponding groups $\rho(A)$ and $\rho(B)$ are conjugate in the group of all permutations of the elements of K if, and only if, (i) A and B are isomorphic, and (ii) A and B have the same index in K.* (18·1)

Proof. Let π be a permutation of the elements of K, and $\pi^{-1}\rho(A)\,\pi = \rho(B)$. Then $\rho(A)$ and $\rho(B)$ are evidently isomorphic, and so then are A and B. If now $k \in K$, $a \in A$ and

$$\pi^{-1}\rho(a)\,\pi = \rho(b),$$

then

$$((k\pi^{-1})\,a)\,\pi = kb;$$

if a here ranges over the whole group A, then b ranges over the whole group B, and the equation shows that the permutation π maps the coset $(k\pi^{-1})A$ of A on to the coset kB of B. Hence π induces a one-to-one mapping of the set of all cosets of A in K on to the set of all cosets of B in K, and A and B have the same index in K.

Conversely, let A and B be isomorphic, and let ϕ be an isomorphic mapping of A on to B; moreover, let A and B have the same index in K. Denote by S and T systems of left coset representatives of A and B respectively in K; then every element $k \in K$ has two unique representations

$$k = sa = tb \quad (s \in S,\ a \in A,\ t \in T,\ b \in B).$$

S and T have the same cardinal, for this is the index of A and also of B in K; let σ be a one-to-one mapping of S on to T. We now define, for every $k = sa \in K$ $(s \in S, a \in A)$,

$$k\pi = s\sigma \,.\, a\phi.$$

Then π is obviously a permutation of the elements of K. If $a^* \in A$ and $k = tb$ $(t \in T, b \in B)$, then

$$\begin{aligned} k\pi^{-1}\rho(a^*)\,\pi &= (t\sigma^{-1} \,.\, b\phi^{-1} \,.\, a^*)\,\pi \\ &= t \,.\, (b\phi^{-1} \,.\, a^*)\,\phi = tb \,.\, a^*\phi \\ &= k\rho(a^*\phi). \end{aligned}$$

As k is an arbitrary element of K, we have thus

$$\pi^{-1}\rho(a^*)\,\pi = \rho(a^*\phi).$$

Letting a^* range over the group A, one obtains

$$\pi^{-1}\rho(A)\,\pi = \rho(B),$$

and the lemma follows. It may be remarked that transformation by π induces on $\rho(A)$ just what corresponds to the isomorphism ϕ of A on to B.

Lemma. *If A and B are isomorphic subgroups of a group G, then there is a supergroup K of G in which A and B have the same index.* (18·2)

Proof. If G is finite, we put $K = G$. If G is infinite, let $K = G \times G'$, where G' denotes an isomorphic copy of G. Let A' and B' correspond to A and B respectively under some fixed isomorphism of G on to G'. Denote by $|G:A|$ the index of A in G, by $|A|$ the order of A, and correspondingly for B, etc. Then

$$K = G \times G' \supseteq G \times B' \supseteq G \supseteq A$$

leads to

$$|K:A| = |G':B'|\,.\,|B'|\,.\,|G:A|,$$

and

$$K = G \times G' \supseteq G \times A' \supseteq G \supseteq B$$

similarly gives

$$|K:B| = |G':A'|\,.\,|A'|\,.\,|G:B|.$$

As $\quad |G':A'| = |G:A|, \quad |G':B'| = |G:B| \quad$ and $\quad |A| = |B|,$

then $|K:A| = |K:B|$, and the lemma follows. One easily sees that the order of K equals that of G. The assumption that A and B are isomorphic can be dispensed with, if Philip Hall's group-theoretical proof (here reproduced) is replaced by a purely set-theoretical one.

Theorem (17·1) now follows if G is first replaced by K and then lemma (18·1) is applied. A further result is obtained from this proof:

Corollary. *If A and B are isomorphic subgroups of a finite group G, then they are conjugate in a finite supergroup H of G.* (18·3)

19. *Miscellaneous applications*

For the purpose of corollary (18·3), H can be chosen as the symmetric group of all permutations of the elements of G. Then the order of H is bounded in terms of that of G. More important is the fact that H then depends on G only, not on A and B; that is, H serves simultaneously for all pairs of isomorphic subgroups of G which one may wish to make conjugate. This is also true for infinite groups G if one takes H to be, for example, the unrestricted symmetric group of all permutations of the elements of $K = G \times G'$, where G' is isomorphic to G. This, and a little more, we also obtain from the following theorem:

Theorem (Higman, Neumann & Neumann 1949). *Let G be a group and let there be given, to every suffix α in a suitable set A, an isomorphism ϕ_α of a subgroup A_α of G on to a subgroup B_α of G. Then there is a group H which contains G as well as a free group T with free generators $t_\alpha (\alpha \in \mathrm{A})$ such that transformation by t_α maps every element $a_\alpha \in A_\alpha$ on to the corresponding element $a_\alpha\phi_\alpha \in B_\alpha$:* (19·1)

$$t_\alpha^{-1} a_\alpha t_\alpha = a_\alpha \phi_\alpha \quad (a \in \mathrm{A},\ a_\alpha \in A_\alpha). \tag{19·2}$$

Proof. Let H_α be the group generated by G and an element t_α and defined by the defining relations of G and

$$t_\alpha^{-1} a_\alpha t_\alpha = a_\alpha \phi_\alpha \quad (\text{for all } a_\alpha \in A_\alpha).$$

We form the free product H of all $H_\alpha (\alpha \in \mathrm{A})$, amalgamating G. Then H is evidently generated by G and all $t_\alpha (\alpha \in \mathrm{A})$, and the relations (19·2) hold in H. To see that the t_α *freely* generate the subgroup T of H which they generate, we apply theorem (4·1) (p. 512); the groups P, G_α, H, A_α, B of the theorem are here H, H_α, G, $\{t_\alpha\}$, $\{1\}$ respectively. It follows from corollary (17·5) that the $\{t_\alpha\}$ are infinite cyclic groups with trivial intersection with G; thus T is the free product of all $\{t_\alpha\}$ $(\alpha \in \mathrm{A})$, and the theorem follows.

Corollary. *Every group G can be embedded in a group G' in which all elements of G of the same order are conjugate.* (19·3)

We take as index set A simply the set of ordered pairs (a, b) of elements of equal order in G, possibly omitting those pairs of elements which are already conjugate in G; and we put

$$A_{(a,b)} = \{a\}, \quad B_{(a,b)} = \{b\},$$
$$a\phi_{(a,b)} = b.$$

If one applies theorem (5·1) (p. 514), observing that G' is obtained from G and infinite cyclic groups by repeatedly forming free products with one amalgamated subgroup and choosing subgroups, then one sees that no new finite numbers appear as orders of elements of G' in addition to those already present in G:

Corollary. *One can choose G' such that every element of finite order in G' is conjugate to an element of G; then G' contains, apart from elements of infinite order, only elements whose orders appear also as orders of elements of G.* (19·4)

Corollary. *If G is denumerable, then so is G'.* (19·5)

For G' is generated by G and elements $t_{(a,b)}$, that is, denumerably many elements.

Theorem. *Every group G can be embedded in a group G^* in which every two elements of equal order are conjugate.* (19·6)

Proof. We form an ascending chain of groups

$$G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots,$$

where we put $G_{i+1} = G_i'$ (cf. corollary (19·3)); and we define G^* as the union of this chain

$$G^* = \cup_i G_i.$$

Given two elements of equal order in G^*, there is a group G_i containing both; then they are conjugate in G_{i+1}, hence also in G^*.

Corollary. *The orders of elements of G^* are, as far as they are finite, also orders of elements of G. If G has elements of n different finite orders, then G^* has exactly $n+1$ classes of conjugate elements. If in particular G is locally infinite, that is $n = 1$, then G^* has only two classes of conjugates: one consists of the unit element, the other of all the remaining elements. In this case G is evidently simple.* (19·7)

Corollary. *If G is denumerable then so is G^*.* (19·8)

For G^* is the union of a denumerable set of denumerable groups. It may be remarked that the only group with only two classes of conjugates and not locally infinite is the cyclic group of order 2; there are only two *periodic* groups (that is groups without elements of infinite order) with exactly three classes of conjugates: the cyclic group of order 3 and the non-Abelian group of order 6.

20. *Embedding a denumerable group in a two-generator group*

The following theorem† answers a question proposed by Kuroš in his book (1944, p. 356).

Theorem. *Every denumerable group G can be embedded in a group H which can be generated by only two elements.* (20·1)

Proof. Let G be generated by elements $g_1, g_2, \ldots$. Without loss of generality we take the generating system (countably) infinite; if G is given in terms of a finite set of generators $g_1, g_2, \ldots, g_k$, we simply add further generators $g_{k+1}, g_{k+2}, \ldots$ without relations, that is, we consider the free product of G and a free group freely generated by $g_{k+1}, g_{k+2}, \ldots$.

Let F_1 denote the free group generated by two elements a, b; in this we choose a subgroup U with infinitely many free generators, one of which is to be b:

$$u_0 = b, \quad u_1 = u_1(a, b), \quad u_2 = u_2(a, b), \ldots. \tag{20·21}$$

Let F_2 denote a further free group with generators b, c; in this we also choose a subgroup V with infinitely many free generators, again with b among them:

$$v_0 = b, \quad v_1 = v_1(b, c), \quad v_2 = v_2(b, c), \ldots. \tag{20·22}$$

Now we form the free product $K = G * F_1$.

This is generated by $a, b, g_1, g_2, \ldots$; if we put $w_0 = b$ and

$$w_i = g_i u_i \quad (i = 1, 2, \ldots),$$

that is

$$g_i = w_i u_i^{-1}, \tag{20·23}$$

then K is also generated by $a, b = w_0, w_1, w_2, \ldots$. Now the elements $w_0, w_1, w_2, \ldots$ are free generators of the subgroup W of K which they generate; for if one maps K homomorphically on to F_1 by putting all elements of G equal to 1, then $w_0, w_1, w_2, \ldots$ are mapped on to the free generators $u_0, u_1, u_2, \ldots$ of U. Hence also the groups W and V are isomorphic, and the mapping ϕ defined by

$$w_i \phi = v_i \quad (i = 0, 1, 2, \ldots)$$

generates an isomorphism. We now form the free product L of K and F_2, amalgamating W and V according to this isomorphism. (This identifies $b \in K$ with $b \in F_2$.) L is generated by $a, w_0, w_1, \ldots, b, c$; but as w_i coincides with v_i in L,

$$w_i = v_i = v_i(b, c) \quad (i = 0, 1, 2, \ldots), \tag{20·24}$$

L is already generated by a, b, c. Now L contains the isomorphic subgroups $F_1 = \{a, b\}$ and $F_2 = \{b, c\}$. We adjoin an element d which transforms F_1 into F_2; thus

$$d^{-1} a d = b, \quad d^{-1} b d = c. \tag{20·25}$$

Denote the resulting group by H. Then H is generated by a, b, c, d, and because of (20·25) also by c and d. Thus G is embedded in a group generated by only two elements. This completes the proof of the theorem.

It is not difficult to express the generators g_i of G in terms of c and d. Equations (20·23) and (20·24) give

$$g_i = v_i u_i^{-1}.$$

† Cf. Higman, Neumann & Neumann (1949); the present proof uses a slightly simpler construction but with the same basic idea.

Here one has to substitute for u_i and v_i their expressions (20·21) and (20·22) in terms of a, b, c, and then to eliminate a and b by means of (20·25). The choice of independent elements u_i in F_1 and v_i in F_2 is highly arbitrary. We may put, for example, by corollary (4·4) (p. 514),

$$u_i = a^{-i}ba^i, \quad v_i = c^{-i}bc^i \quad (i = 0, 1, 2, \ldots).$$

Then we obtain the generators of G in the explicit form

$$g_i = c^{-i}dcd^{-1}c^id^2c^{-i}d^{-1}c^{-1}dc^id^{-2}. \tag{20·3}$$

Corollary. *If G can be defined by n defining relations, then H can also be chosen as a group with n defining relations.* (20·4)

For if
$$r_1(g_1, g_2, \ldots) = 1, \ldots, r_n(g_1, g_2, \ldots) = 1 \tag{20·5}$$
are the defining relations of G, then the generators c and d of H only have to be made to satisfy the n relations obtained by substituting the expressions (20·3) for the g_i in (20·5).

It may be added without proof that every denumerable group can even be embedded in a group with two generators of finite order; specifically one generator can be given the order 2 (or any greater number), the other the order 8 (or any greater number). It is an unsolved problem whether one can do it even with two generators of orders 2 and 3 respectively, in other words whether every denumerable group can be embedded in a factor group of the modular group.

The theorem can be put into a different form:

Theorem. *Let F be the free group generated by two elements c, d; denote by E the subgroup of F generated by the elements*

$$e_i = c^{-i}dcd^{-1}c^id^2c^{-i}d^{-1}c^{-1}dc^id^{-2} \quad (i = 1, 2, \ldots).$$

If R is an arbitrary normal subgroup of E, and if R^F denotes the normal closure of R in F (that is, the least normal subgroup of F containing R), then

$$R^F \cap E = R. \tag{20·6}$$

Proof. We put $E/R = G$ and apply the construction by means of which we have proved theorem (20·1). Then $H = F/R^F$. That G is a subgroup of H means

$$E \cup R^F/R^F \simeq E/R.$$

Hence
$$E/E \cap R^F \simeq E/R,$$

an isomorphism being induced by the identical mapping of E. As, moreover, $R \subseteq R^F \cap E$, it follows that $R = R^F \cap E$, and the theorem is proved.

The relationship between E and F expressed by this theorem can be put differently:

(i) Every normal subgroup of E is the intersection of E and a normal subgroup of F.

(ii) Every homomorphism of E on to a group G can be extended to a homomorphism of F on to a supergroup H of G.

Let us call a subgroup E of a—not necessarily free—group F an *E-subgroup* if it has this relation to F; then it is easy to see that, for example, free factors, direct factors, simple subgroups and subgroups of the centre of a group F are always E-subgroups of F. The relation is, moreover, transitive: If E is an E-subgroup of F and D an E-subgroup of E, then D is also an E-subgroup of F. The point of theorem (20·6) is that the free group with two generators possesses E-subgroups with infinitely many free generators.

Mal'cev (cf. Kuroš 1944, p. 357) has proposed the question whether there is a denumerable 'universal embedding group', that is, a denumerable group containing an isomorphic copy of every denumerable group as a subgroup. This question can also be answered: There cannot be such a group. For a denumerable group has only denumerably many two-generator subgroups; but there are continuously many non-isomorphic two-generator groups (B. H. Neumann 1937). Hence a group which is to contain an isomorphic copy of every denumerable group must have order at least equal to the cardinal of the continuum. The unrestricted symmetric group of permutations of a denumerably infinite set provides an obvious example.

Theorem (20·1) is in its way a best possible result; for a non-denumerable group requires as many generators as it has elements, and, on the other hand, a non-cyclic group cannot be embedded in a group with only one generator. We can, however, derive a related result for non-denumerable groups:

THEOREM. *Every group G can be embedded in a group G^* in which every denumerable subgroup (or, what amounts to the same, every denumerable subset) is contained in a two-generator subgroup.* (20·7)

If G is denumerable, then this theorem contains no more than theorem (20·1). For its proof we require a lemma.

LEMMA. *Every group G can be embedded in a group G' in which every denumerable subgroup of G is contained in a two-generator subgroup.* (20·8)

Proof. Denoting the denumerable subgroups of G by K_α, where α ranges over a suitable index set A, we can (by theorem (20·1)) embed every group K_α in a group H_α with two generators. By theorem (15·4) (p. 532) there is then a group G' containing G and all H_α (here G, K_α, H_α, G' play the roles of Q, K_α, G_α, P in the theorem referred to). G' obviously has the desired property.

Proof of theorem (20·7). We form a well-ordered ascending chain of groups

$$G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\omega \subseteq G_{\omega+1} \subseteq \ldots \subseteq G_{\omega_1} = G^* \tag{20·9}$$

as follows. If λ is a denumerable ordinal and if G_λ has already been defined, we put $G_{\lambda+1} = G'_\lambda$ as defined in the lemma. If λ is a denumerable limit ordinal or ω_1, the least non-denumerable ordinal, we put

$$G_\lambda = \bigcup_{\mu<\lambda} G_\mu.$$

Now let $g_1, g_2, \ldots$ be an arbitrary denumerable subset of $G^* = G_{\omega_1}$; let $G_{\lambda(n)}$ be the first group of the sequence (20·9) which contains $g_1, g_2, \ldots, g_n$. Then $\lambda(n) < \omega_1$ for every n, and the least ordinal λ not less than all $\lambda(n)$,

$$\lambda = \text{l.u.b.}\,\lambda(n),$$

is also still denumerable. G_λ contains the whole subgroup $\{g_1, g_2, \ldots\}$, and in $G_{\lambda+1}$, and thus also in G^*, this is then contained in a two-generator subgroup. This proves the theorem.

One easily confirms incidentally that in G_ω every finite subset is contained in a two-generator subgroup. A group with this property is said† to have *rank* 2; thus every group can be embedded in a group of rank 2.

† Cf. Kurosch (1939 or 1944, §48); Mal'cev (1948).

Chapter V. Three remarkable groups of Graham Higman's

21. *Two problems of Hopf's*

More than twenty years ago Heinz Hopf proposed two group-theoretical problems (and a topological one, from which they had stemmed). The problem which was known as 'Hopf's problem' (cf. Kuroš 1944, pp. 75, 356) asks whether a finitely generated group can be isomorphic to a proper factor group of itself; the other problem, which Hopf recalled at the recent Fourth British Mathematical Colloquium, asks whether two finitely generated groups are necessarily isomorphic if they are homomorphic images of each other. Both problems have now been solved; a group with two generators and infinitely many defining relations has been constructed by the author (1950), another with three generators and only two defining relations by Graham Higman (1951 *a*), each of these groups being isomorphic to a proper factor group of itself. Higman's example is given here, but immediately in a form in which the second Hopf problem is also solved (B. H. Neumann 1953).

Let G be the group with three generators a, b, c and the two defining relations

$$a^{-1}ba = b^2, \tag{21·1}$$

$$bc = cb. \tag{21·2}$$

The element $$b_1 = aba^{-1}$$

is a square root of b, that is to say, $b_1^2 = b$. We denote by d the commutator of b_1 and c,

$$d = b_1^{-1}c^{-1}b_1c,$$

and first show that d is not the unit element in G.

G is the free product of the subgroup A generated by a and b and defined by (21·1), and the subgroup C generated by b and c and defined by (21·2), with the cyclic subgroup $B = \{b\}$ amalgamated. Neither b_1 nor c lie in this amalgamated subgroup. Hence in the commutator $b_1^{-1}c^{-1}b_1c$ no two successive factors belong to the same group A or C; by corollary (3·2) (p. 511) then this element has length 4 and is certainly not the unit element.

Let D denote the normal closure of d in G,

$$D = \{d\}^G,$$

that is, the least normal subgroup of G containing d. Then D is not trivial, as we have seen; but D has trivial intersection with both A and C. To see this one only has to consider the homomorphism of G on to the direct product of A and the cycle $\{c\}$; as this direct product contains A and C, the kernel of this homomorphism has trivial intersection with both subgroups; and this kernel contains D.

Now it follows from the structure theory of subgroups of generalized free products† that if a normal subgroup of a free product with one amalgamated subgroup has trivial intersection with the factors, it is a free group. Hence D is a non-trivial free group.

Let S denote the subgroup of D generated by all squares of elements of D. Then S is a proper subgroup of D. If next T stands for the normal closure of d^2 in G,

$$T = \{d^2\}^G,$$

† Hanna Neumann (1949, theorem 13·0 and the first paragraph of p. 540).

then T is a subgroup of S, because it is generated by the conjugates of d^2, that is, by squares of elements of D. Thus T is a normal subgroup of G properly contained in D. In particular d cannot lie in T; but d^2 does. Putting

$$H = G/T,$$

we see that the element dT of H is not the unit element but has order 2.

On the other hand, G is locally infinite, being the free product of the locally infinite groups A and C with an amalgamated subgroup (cf. corollary (5·5), p. 515). Hence G and H are not isomorphic.

Evidently H is a homomorphic image of G. We now show that G is also a homomorphic image of H. Distinguishing the elements of H by primes:

$$aT = a', \quad bT = b', \quad cT = c'$$

and correspondingly $$b_1' = a'b'a'^{-1}, \quad d' = b_1'^{-1}c'^{-1}b_1'c',$$

the defining relations of H are
$$a'^{-1}b'a' = b'^2, \tag{21·3}$$
$$b'c' = c'b', \tag{21·4}$$
$$d'^2 = 1. \tag{21·5}$$

We consider the mapping μ of H into G defined by

$$a'\mu = a, \quad b'\mu = b^2, \quad c'\mu = c.$$

As in G obviously $$a^{-1}b^2a = b^4 = (b^2)^2,$$
$$b^2c = cb^2,$$

the relations corresponding to (21·3) and (21·4) are satisfied by the maps under μ. Also

$$b_1'\mu = ab^2a^{-1} = b$$

and $$d'\mu = (b_1'^{-1}c'^{-1}b_1'c')\,\mu = b^{-1}c^{-1}bc = 1.$$

Hence the relation corresponding to (21·5) is also satisfied, even as a consequence of the more stringent $d'\mu = 1$. The mapping μ then generates a homomorphism of H into G; but this is in fact on to G, because a, b^2, c generate the whole of G. Thus we have shown:

THEOREM. *The groups G and H defined above in terms of three generators with two and three defining relations, respectively, are not isomorphic, but are homomorphic maps of each other.* (21·6)

If we combine the homorphisms of G on to H and of H on to G, we obtain an endomorphism of G on to itself. This maps a on to a, b on to b^2, c on to c; its kernel is $D = \{d\}^G$. Thus the factor group $$G_1 = G/D$$

is isomorphic to G; but it is a *proper* factor group because D is not trivial. Thus we also see:

COROLLARY. *The group G defined above in terms of three generators with two defining relations is isomorphic to a proper factor group of itself.* (21·7)

22. *Chains of normal closures and subgroups in a free group*

If we put, still in the notation of §21,

$$a^{-1}bab^{-2} = r, \quad b^{-1}c^{-1}bc = s,$$

then G is isomorphic to the factor group F/R of the free group $F = \{a, b, c\}$ with respect to the normal closure of r and s $$R = \{r, s\}^F.$$

Similarly, putting again $$ab^{-1}a^{-1}c^{-1}aba^{-1}c = d,$$

G_1 is isomorphic to the factor group F/R_1, where

$$R_1 = \{r, s, d\}^F = \{r, d\}^F.$$

To see that s is contained in $\{r, d\}^F$ we only have to observe that if r and d are equated to 1, the square root $b_1 = aba^{-1}$ of b commutes with c, hence b commutes with c, thus s also equals 1.

R_1 is a proper supergroup of R, as G_1 is a proper factor group of G. Now G is mapped isomorphically on to G_1, and thus R on to R_1, if we replace a by a, b by $b_1 = aba^{-1}$, and c by c; for this replaces s by d, and r by

$$r_1 = a^{-1}.aba^{-1}.a.(aba^{-1})^{-2} = bab^{-2}a^{-1},$$

which is a conjugate of r. Hence we obtain R_1 from R by applying to the free group F the automorphism α defined by

$$a\alpha = a,$$
$$b\alpha = aba^{-1},$$
$$c\alpha = c.$$

Thus $$R \subset R_1 = R\alpha.$$

Repeated application of α gives a properly ascending chain† of subgroups

$$R \subset R\alpha \subset R\alpha^2 \subset \dots,$$

each term of which is normal in F:

$$R\alpha^n = \{r\alpha^n, s\alpha^n\}^F;$$

every term then is the normal closure of two elements. We deduce two consequences, the first of which answers another question proposed by Kuroš (1944, p. 357):

THEOREM. *There exists a normal subgroup of a free group of finite rank (in our case: three) which is mapped on to a proper supergroup (or, equivalently, a proper subgroup) of itself by an automorphism of the free group.* (22·1)

THEOREM. *There exists a properly ascending infinite chain of normal subgroups of a free group of finite rank (in our case: three), each term of the chain being the normal closure of two elements.* (22·2)

In both cases the rank of the free group can be depressed from 3 to 2. For a normal subgroup mapped on to a proper supergroup by an automorphism one uses the author's example (1950); the normal subgroup is then, however, not the normal closure of a finite set of elements. A properly ascending chain of normal closures of two elements in a free group of rank 2 one obtains from the example here presented by applying theorem (20·6) (p. 541); however, the normal subgroups in the chain then are no longer obtained from each other by automorphisms of the free group. It is not known at present whether the free group of rank 2 contains a normal subgroup which is the normal closure of a finite set of elements and which is mapped on to a proper supergroup by an automorphism of the free group. It is likewise unknown whether there is a group with two generators and finitely many defining relations which is isomorphic to one of its own proper factor groups.

† By repeated application of the inverse automorphism the chain can also be completed to a properly descending infinite chain.

One can ask the corresponding questions for subgroups generated by finite sets of elements instead of for normal closures of finite sets of elements: can there exist in a finitely generated free group F a properly ascending chain of subgroups of fixed finite rank? Can a subgroup of finite rank in a free group F be mapped on to a proper supergroup of itself by an automorphism of F? The answer to both questions is negative; and that independently of the rank of F. This results from the following theorem.

THEOREM. *Let the group G be the union of a chain of proper subgroups*

$$G_1 \subset G_2 \subset G_3 \subset \dots$$

each of which can be generated by n or fewer elements (n a finite number). If G can be decomposed into a free product

$$G = H * K,$$

and if H can be finitely generated, then H can be generated by fewer than n elements, and K cannot be finitely generated. (22·3)

Proof. As union of an infinite chain of proper subgroups G can certainly not be finitely generated.† Hence the same is true of at least one of the two free factors. Let now H have a finite set of generators; then there is a number i such that they all lie in G_i, that is $H \subseteq G_i$. Then G_i is also freely decomposable,‡ with H as one of the free factors:

$$G_i = H * K_i.$$

By Gruschko's theorem§ the minimal number of generators of G_i is the sum of the corresponding minimal numbers for H and K_i. We may assume K_i non-trivial, because we can, if need be, replace G_i by a proper supergroup which occurs in the ascending chain. The minimal number of generators of H is, therefore, actually less than n, and the theorem is proved.

COROLLARY. *Under the same assumptions G is not free.* (22·4)

For a free group of infinite rank has free factors of arbitrarily high finite rank.

COROLLARY (Takahasi 1950; Higman 1951 *a*). *A properly ascending chain of subgroups of finite, bounded rank in a free group breaks off.* (22·5)

For otherwise its union would not be free, but every subgroup of a free group is free.

COROLLARY. (Higman 1951 *a*). *No subgroup of finite rank of a free group is mapped on to a proper supergroup by an automorphism of the free group.* (22·6)

23. *Finitely generated infinite simple groups*

We next turn to the question of the existence of finitely generated infinite simple groups. This question was raised by Kuroš (1944, p. 356) and answered by Higman (1951 *b*), whose existence proof we here reproduce.

† B. H. Neumann (1937); cf. also Kuroš (1944, p. 78).
‡ Kurosch (1934); cf. also (1944, §44).
§ Gruschko (1940); cf. also B. H. Neumann (1943 *b*) and Kuroš (1944, §46).

The group G which we shall study is generated by four elements a, b, c, d and defined by the relations

$$a^{-1}ba = b^2, \tag{23·1}$$

$$b^{-1}cb = c^2, \tag{23·2}$$

$$c^{-1}dc = d^2, \tag{23·3}$$

$$d^{-1}ad = a^2. \tag{23·4}$$

We first show that G is an infinite group.

In the group A with two generators a, b and the defining relation (23·1), the cyclic subgroups generated by a and b are infinite and have trivial intersection. If B denotes the group generated by b, c with defining relation (23·2), then b and c similarly generate infinite cycles with trivial intersection. We can form the free product of A and B amalgamating b and thus obtain a group P generated by a, b, c and defined by relations (23·1) and (23·2). The subgroup of P generated by a and c is the free group freely generated by these two elements (Theorem (4·1), p. 512). The group Q generated by c, d, a with defining relations (23·3) and (23·4) is evidently isomorphic to P; hence also in this group a and c generate a free group of rank 2. If now we form the free product of P and Q amalgamating the free subgroup generated by a and c, we obtain just G; and as, for example, $\{a\}$ is an infinite cycle, G is certainly infinite.

Next we show that G has no proper subgroup of finite index. If G had a subgroup of finite index >1, then it would also have a normal subgroup N of finite index >1; then G/N would be a finite group G_1 with four generators, which we again denote by a, b, c, d and which satisfy *inter alia* the relations (23·1 to 23·4). We show that such a group G_1 is necessarily trivial, thus $N = G$, contrary to the assumption that the index of N in G is greater than 1.

We even show a little more. We add to the defining relations of G only one further relation

$$a^{\alpha} = 1 \quad (\alpha > 0); \tag{23·5}$$

such a relation must certainly be valid in a finite group; then it will be seen that the resulting group—let us again denote it by G_1—is trivial.

Transforming b by a^{α} we obtain from (23·1) and (23·5)

$$b = b^{2^{\alpha}}.$$

Hence b also has finite order, and the order β, say, of b divides $2^{\alpha}-1$. Similarly we see that c has finite order γ which divides $2^{\beta}-1$, and that d has finite order δ which divides $2^{\gamma}-1$; and finally α must also divide $2^{\delta}-1$. If one of the numbers α, β, γ, δ equals 1, then they all equal 1, and G_1 is trivial. Let us then assume on the contrary that $\alpha, \beta, \gamma, \delta$ are all greater than 1; if we denote the least prime divisor of $\alpha, \beta, \gamma, \delta$ by $\pi_{\alpha}, \pi_{\beta}, \pi_{\gamma}, \pi_{\delta}$ respectively, then

$$\pi_{\alpha} < \pi_{\beta}.$$

For as β divides $2^{\alpha}-1$, we have $$2^{\alpha} \equiv 1 \pmod{\pi_{\beta}};$$

hence α is divisible by the exponent to which 2 belongs modulo π_{β}, that is by a divisor (>1) of $\pi_{\beta}-1$. Similarly we get

$$\pi_{\beta} < \pi_{\gamma} < \pi_{\delta} < \pi_{\alpha},$$

thus finally $\pi_{\alpha} < \pi_{\alpha}$, which shows that the assumption that $\alpha, \beta, \gamma, \delta$ are greater than 1 leads to a contradiction.

Thus we have shown that G has no proper normal subgroup of finite index, hence no proper subgroup whatever of finite index. As G is finitely generated it possesses a maximal proper normal subgroup N (B. H. Neumann 1937). Then G/N is a simple group because N is a maximal normal subgroup, and infinite because N is a proper subgroup of G, and finitely generated because G is finitely generated. This shows the *existence* of an infinite simple group with finitely many generators. This proof is remarkable in that it is a non-constructive existence proof: such proofs are not uncommon in analysis, but they rarely occur in algebra and theory of groups. Recently Ruth Camm has constructed explicitly a simple group† which is the free product of two free groups of rank 2 with an amalgamated subgroup; this can then be generated by four elements, and in fact also by only two, and requires infinitely many defining relations. It is an unsolved problem whether there are infinite simple groups with a finite number of defining relations.‡

One can vary Higman's construction by taking more than four generators with correspondingly more relations, that is by considering the group with generators $a_1, a_2, \ldots, a_n$ and defining relations

$$a_i^{-1} a_{i+1} a_i = a_{i+1}^2 \quad (i = 1, 2, \ldots, n-1),$$

$$a_n^{-1} a_1 a_n = a_1^2.$$

The same way leads to the same goal. If, however, one puts $n<4$, then one only gets the trivial group. This is immediate for $n = 1, 2$. If $n = 3$, one can proceed, for example, as follows:

G has three generators a, b, c and defining relations

$$a^{-1}ba = b^2,$$

$$b^{-1}cb = c^2,$$

$$c^{-1}ac = a^2.$$

The first gives $$b^{-1}ab = ab^{-1},$$

hence $$b^{-i}ab^i = ab^{-i},$$

and similarly from the second relation,

$$c^{-i}bc^i = bc^{-i}.$$

Transforming the third relation by b, the left-hand side becomes

$$b^{-1}c^{-1}acb = c^{-2}ab^{-1}c^2 = a^4c^2b^{-1},$$

and the right-hand side $$b^{-1}a^2b = (ab^{-1})^2 = a^2b^{-3}.$$

Hence $$c^2 = a^{-2}b^{-2}.$$

Now transforming b by c^{-2}, we get

$$bc^2 = c^2bc^{-2} = a^{-2}b^{-2}\,.\,b\,.\,b^2a^2 = b^4;$$

thus $c^2 = b^3$, and b and c^2 commute. But then $c = c^2$, $c = 1$, and also $a = b = 1$.

† Ruth Camm (1953); the method yields continuously many such groups.

‡ Higman (1951 *b*). Such a group, if it exists, can necessarily be finitely generated.

Incidentally there results a new example of a system of three groups whose generalized free product with certain amalgamations does not exist (cf. chapter II, § 11). For if we put

$$G_1 = \{a_1, b_1;\ a_1^{-1}b_1a_1 = b_1^2\},$$
$$G_2 = \{b_2, c_2;\ b_2^{-1}c_2b_2 = c_2^2\},$$
$$G_3 = \{c_3, a_3;\ c_3^{-1}a_3c_3 = a_3^2\},$$
$$H_{12} = \{b_1\}, \quad H_{21} = \{b_2\},$$
$$H_{13} = \{a_1\}, \quad H_{31} = \{a_3\},$$
$$H_{23} = \{c_2\}, \quad H_{32} = \{c_3\},$$

(with the obvious isomorphisms between these subgroups), then the group P' of chapter II (the canonic group) becomes trivial, and the generalized free product does not exist. Here the isomorphism and strong intersection properties are lacking, the weak intersection property (trivially) not. This example may be compared with that in chapter I, § 5, where very similar relations lead to a quite different result.

24. *A problem of Kurosch's*

Finally, we deal with another question proposed by Kuroš (1944, p. 358): whether there exists a group which is not free but every denumerable subgroup of which is free; we shall present an example of such a group which is again due to Graham Higman (1951 *c*). The corresponding question for free *Abelian* groups has been solved by Specker† by showing that the unrestricted direct product of a denumerable infinity of infinite cyclic groups is not a free Abelian group (that is to say, not the restricted direct product of continuously many infinite cyclic groups), but that every denumerable subgroup of it is a free Abelian group (that is then the restricted direct product of at most denumerably many infinite cyclic groups).

Before proceeding to the construction of Higman's group we prove a theorem likewise due to Higman:‡

THEOREM. *Let the group G contain a family $\mathfrak{F}$ of subgroups F_α (where α ranges over an index set* A) *with the following properties:*

(*a*) *Every F_α is a free group of finite rank.*
(*b*) *If $F_\alpha \subset F_\beta$ then F_α is a free factor of F_β.*
(*c*) *Every finite subset of G (or, equivalently, every finitely generated subgroup of G) is contained in a group F_α.*

Then every denumerable subgroup of G is free. (24·1)

Proof. One sees easily that G is locally free. If H is a denumerable subgroup of G and if H is finitely generated, then H is a subgroup of a free group F_α, hence itself free. There remains only the case that H requires infinitely many generators $h_1, h_2, \ldots$. We form a chain of groups $F_{\alpha(1)} \subseteq F_{\alpha(2)} \subseteq \ldots$ in $\mathfrak{F}$ as follows. Let $F_{\alpha(1)}$ be an arbitrary group in $\mathfrak{F}$ containing h_1;

† Specker (1950); the result is also implicit in more general results of Baer (1937).

‡ Higman (1951 *c*); the reader is referred to this paper also for a number of further interesting results, and for a more general theorem of which the one here proved is only a part. For some of the results, cf. also Takahasi (1950).

and inductively, if $F_{\alpha(i)}$ is already defined, then $\{F_{\alpha(i)}, h_{i+1}\}$ is finitely generated, hence contained in a group of the family $\mathfrak{F}$, and one such group we take as $F_{\alpha(i+1)}$. As $F_{\alpha(i)}$ is a free factor of $F_{\alpha(i+1)}$, and as both are free groups, a system of free generators of $F_{\alpha(i)}$ can be completed to a system of free generators of $F_{\alpha(i+1)}$. Starting from free generators of $F_{\alpha(1)}$ we thus obtain step by step a system of free generators of the union F of the chain†

$$F_{\alpha(1)} \subseteq F_{\alpha(2)} \subseteq \dots.$$

Hence F is a free group; but H is evidently contained in F, and is, therefore, itself free.

Corollary. *With the same assumption every F_α is a free factor of every finitely generated subgroup F of G which contains F_α.* (24·2)

For F is contained in a group $F_\beta \in \mathfrak{F}$; F_α is by assumption a free factor of F_β. Because of $F_\alpha \subseteq F \subseteq F_\beta$, then F_α is also a free factor of F (Kurosch 1934; cf. also 1944, §44).

Now we define Higman's group G^* by transfinite induction. Let G_0 be a free group with $\aleph_0$ generators $a_{01}, a_{02}, a_{03}, \dots$. If λ is a denumerable ordinal, and if G_λ is a free group with $\aleph_0$ free generators $a_{\lambda 1}, a_{\lambda 2}, a_{\lambda 3}, \dots$, then we take for $G_{\lambda+1}$ the group freely generated by elements

$$a_{\lambda+1,1}, \quad a_{\lambda+1,2}, \quad a_{\lambda+1,3}, \dots$$

and identify G_λ with a subgroup of $G_{\lambda+1}$ by putting

$$a_{\lambda,i} = a_{\lambda+1,i}\, a_{\lambda+1,i+1}^{-2} \quad (i = 1, 2, \dots). \tag{24·3}$$

If λ is a denumerable limit ordinal or if $\lambda = \omega_1$ (that is the least non-denumerable ordinal), and if G_μ has already been defined for all $\mu < \lambda$, then we take as G_λ the union

$$G_\lambda = \bigcup_{\mu<\lambda} G_\mu.$$

We thus evidently define a well-ordered ascending chain of groups

$$G_0 \subset G_1 \subset G_2 \subset \dots \subset G_\kappa, \tag{24·4}$$

which can only break off when G_κ is no longer a free group of rank $\aleph_0$. This can certainly happen only if κ is a limit ordinal, as the definition immediately shows; and it must happen at the latest for $\kappa = \omega_1$, for the chain (24·4) is properly ascending, and G_{ω_1}—provided the chain can be carried thus far—is the union of $\aleph_1$ denumerable groups, hence itself of order $\aleph_1$ and can no longer be generated by $\aleph_0$ elements. Our first aim is to show that the chain (24·4) can in fact be continued that far, in other words, that for every *denumerable* limit ordinal λ the union

$$G_\lambda = \bigcup_{\mu<\lambda} G_\mu$$

is still a free group of rank $\aleph_0$, provided all G_μ with $\mu < \lambda$ are free groups of rank $\aleph_0$.

G_λ is clearly still denumerable, being a union of denumerably many denumerable groups. As union of a properly ascending chain, G_λ can certainly not be finitely generated.‡ It only remains to show that G_λ is free. This we show by transfinite induction, using theorem (24·1); to this end we sharpen both the induction hypothesis and the proposition to be proved.

† The chain need not increase properly, and the union can itself belong to the chain and the family; the conclusion is not affected.

‡ B. H. Neumann (1937); cf. also Kuroš (1944, p. 78).

Let Φ_μ be the family of groups

$$F_{\mu i} = \{a_{\mu 1}, a_{\mu 2}, \ldots, a_{\mu i}\},$$

where μ is an ordinal for which G_μ is already defined and is a free group with free generators $a_{\mu 1}, a_{\mu 2}, a_{\mu 3}, \ldots$, and where i ranges over the positive integers. Further put

$$\mathfrak{F}_\mu = \bigcup_{\nu \leqslant \mu} \Phi_\nu;$$

and if G_μ is already defined for all $\mu < \lambda$, put

$$\mathfrak{F}^*_\lambda = \bigcup_{\mu<\lambda} \Phi_\mu = \bigcup_{\mu<\lambda} \mathfrak{F}_\mu.$$

We next show that these families satisfy the assumptions of theorem (24·1):

(a_μ) Every group $F_{\nu i}$ is evidently a free group of finite rank.

We now assume further

(b_μ) that $\nu \leqslant \mu$, $\pi \leqslant \mu$ and $F_{\nu i} \subset F_{\pi j}$ implies that $F_{\nu i}$ is a free factor of $F_{\pi j}$, and

(c_μ) that every finitely generated subgroup of G_μ is contained in a group $F_{\nu i} \in \mathfrak{F}_\mu$.

We first show the corresponding properties of $\mathfrak{F}_{\mu+1} = \mathfrak{F}_\mu \cup \Phi_{\mu+1}$; here $(c_{\mu+1})$ is immediately obvious, because the generators of a finitely generated subgroup of $G_{\mu+1}$ involve only a finite number of the $a_{\mu+1,i}$, and the subgroup then is contained in a $F_{\mu+1,i} \in \Phi_{\mu+1}$. To prove also $(b_{\mu+1})$ we remark first that the case $\nu \leqslant \mu$, $\pi \leqslant \mu$ is already contained in (b_μ). If $\nu = \mu+1$, $\pi \leqslant \mu$ then $a_{\mu+1,1}$ is an element of $F_{\mu+1,i} = F_{\nu i}$, but not of $F_{\pi j}$, and the premiss is not satisfied. If $\nu = \mu+1$, $\pi = \mu+1$, the validity of the proposition follows immediately from the definition of the groups $F_{\nu i}$. Now one easily sees from equations (24·3) that

$$a_{\mu 1}, \quad a_{\mu 2}, \quad \ldots, \quad a_{\mu i}, \quad a_{\mu+1,i+1}, \quad a_{\mu+1,i+2}, \quad \ldots$$

form also a system of free generators of $G_{\mu+1}$. Hence $F_{\mu i}$ is a free factor of $F_{\mu+1,j}$ if $F_{\mu i} \subset F_{\mu+1,j}$, that is if $i<j$. Finally if $\nu<\mu$, $\pi = \mu+1$, then $F_{\nu i}$ is contained in, and a free factor of, a group $F_{\mu p}$, and $F_{\mu p}$ is contained in, and a free factor of, a group $F_{\mu+1,q}$, and thus $F_{\nu i}$ is in its turn a free factor of $F_{\mu+1,q}$. $F_{\pi j} = F_{\mu+1,j}$ is either a subgroup and free factor or a supergroup of $F_{\mu+1,q}$, and one again sees easily that $F_{\nu i}$ is also a free factor of $F_{\pi j}$. This completes the proof of $(b_{\mu+1})$; $(a_{\mu+1})$ is obvious.

Next let λ be a limit ordinal (denumerable or ω_1), and let the induction hypothesis be satisfied for all $\mu<\lambda$. Then $\mathfrak{F}^*_\lambda$ satisfies conditions (a^*_λ) trivially; and also (b^*_λ): because $\nu<\lambda$, $\pi<\lambda$ and $F_{\nu i} \subset F_{\pi j}$ implies that $F_{\nu i}$ is a free factor of $F_{\pi j}$, for there is a $\mu<\lambda$ with $\nu \leqslant \mu$, $\pi \leqslant \mu$. If, furthermore, a finitely generated subgroup H of $G_\lambda = \bigcup_{\mu<\lambda} G_\mu$ is given, then H is already contained in a G_μ with $\mu<\lambda$, hence in a suitable $F_{\nu i} \in \mathfrak{F}_\mu$. Thus we have also (c^*_λ): every finitely generated subgroup of G_λ is contained in a group $F_{\nu i} \in \mathfrak{F}^*_\lambda$. Now by theorem (24·1) every denumerable subgroup of G_λ is free, and thus G_λ is itself free if λ is denumerable. We finally show that then

$$\mathfrak{F}_\lambda = \mathfrak{F}^*_\lambda \cup \Phi_\lambda$$

again satisfies the assumptions (a_λ), (b_λ), (c_λ), where Φ_λ is defined from an arbitrary system of free generators $a_{\lambda 1}, a_{\lambda 2}, a_{\lambda 3}, \ldots$ of G_λ. Again (a_λ) is obvious, and (c_λ) is a trivial consequence of (c^*_λ). To prove (b_λ) we need only consider the case $\nu<\lambda$, $\pi = \lambda$, $F_{\nu i} \subset F_{\lambda j}$. But then $F_{\nu i}$ is a free factor of $F_{\lambda j}$ by the corollary (24·2); for $F_{\nu i}$ belongs to the family $\mathfrak{F}^*_\lambda$ which satisfies the assumptions of theorem (24·1), and $F_{\lambda j}$ is finitely generated.

Transfinite induction thus shows that G_λ is defined for all denumerable λ and that it is a free group of rank $\aleph_0$, and that G_{ω_1} is at any rate still defined; this latter group we denote by G^*. By the argument just presented the family $\mathfrak{F}^*_{\omega_1}$ satisfies in G^* the conditions of theorem (24·1); hence every denumerable subgroup of G^* is free. It only remains to prove that G^* is not itself free. Let us assume on the contrary that G^* is a free group with free generators a^*_α, where α ranges over an index set A; this has the cardinal $\aleph_1$, for we had already seen that the cardinal of G^* is $\aleph_1$. Now let the generators of G_0 be expressed in terms of the a^*_α; in every $a_{0i} \in G_0$ only a finite number of the a^*_α is involved, hence for all of them together only denumerably many a^*_α, and there are still $\aleph_1$ further free generators of G^* left over. If N denotes the normal closure in G^* of all those a^*_α which enter the expressions for $a_{01}, a_{02}, a_{03}, \ldots$, then we find that G^*/N is still a free group of rank $\aleph_1$, in particular, therefore, a non-trivial free group.

But we now show inductively that N contains all groups G_λ of the chain which defines G^*; for assume $G_\mu \subseteq N$ has already been established. Because of equations (24·3) then

$$Na_{\mu+1,i} = Na^2_{\mu+1,i+1} \quad (i = 1, 2, \ldots);$$

this means that the element $Na_{\mu+1,i}$ of G/N is a 2^n-th power for arbitrarily large positive integers n. In a free group only the unit element has this property; hence

$$a_{\mu+1,i} \in N \quad (i = 1, 2, \ldots),$$

which implies $G_{\mu+1} \subseteq N$. If λ is a limit ordinal and if $G_\mu \subseteq N$ is true for all $\mu < \lambda$, then also

$$G_\lambda = \bigcup_{\mu<\lambda} G_\mu \subseteq N.$$

Transfinite induction then shows $G^* \subseteq N$, that is $N = G^*$, contrary to G^*/N being a non-trivial free group. Thus the assumption that G^* is free leads to a contradiction, and we have proved:

THEOREM. *The group G^* constructed above is not free although all its denumerable subgroups are free.* (24·5)

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